Strength of Materials
(Problems and Solutions)

Editor:

Tamás Mankovits PhD

Dávid Huri

The course material/work/publication is supported by the EFOP-3.4.3-16-2016-00021 "Development of the University of Debrecen for the Simultaneous Improvement of Higher Education and its Accessibility" project. The project is supported by the European Union and co-financed by the European Social Fund.

Editor:

Tamás Mankovits PhD

Dávid Huri

Authors:

Tamás Mankovits PhD

Dávid Huri

Manuscript closed: 18/04/2018

ISBN 978-963-490-030-6

Published by the University of Debrecen

TABLE OF CONTENTS

INTRODUCTION

During the subject of Mechanics of Materials (Strength of Materials) the investigations are limited only the theory of linear elasticity.

The elasticity is the mechanics of elastic bodies. It is known that solid bodies change their shape under load. The elastic body is capable to deform elastically. The elastic deformation means that the body undergone deformation gets back its original shape once the load ends. The task of the elasticity is to determine the displacement, the strain and the stress states of the body points. Depending on the connection between the stress and strain this elastic deformation can be linear or nonlinear. If the function between the stress and strain is linear we say linear elastic deformation. Materials which behave as linear are the steel, cast iron, aluminum, etc. If the function between the stress and strain is nonlinear we say nonlinear elastic deformation. The most typical nonlinear material is the rubber. This exercise book deals with only the theory of linear elasticity.

The theory of elasticity establishes a mathematical model of the problem which requires mathematical knowledge to be able to understand the formulations and the solution procedures. The governing partial differential equations are formulated in vector and tensor notation.

SZÉCHENYI 2020

1. DISPLACEMENT STATE OF SOLID BODIES

1.1. Theoretical background of the displacement state of solid bodies

Displacement field

Considering the theory of linear elasticity the deformed body under loading gets back its original shape once the loading ends.

Now, let us consider a general elastic body undergone deformation as shown in Figure 1.1. As a result of the applied loadings, the elastic solids will change shape or deform, and these deformations can be quantified with the displacements of material points in the body. The continuum hypothesis establishes a displacement field at all points within the elastic solid *[1].*

We have selected two arbitrary points in the body P and Q. In the deformed state, points P and Q move to point P' and Q'.

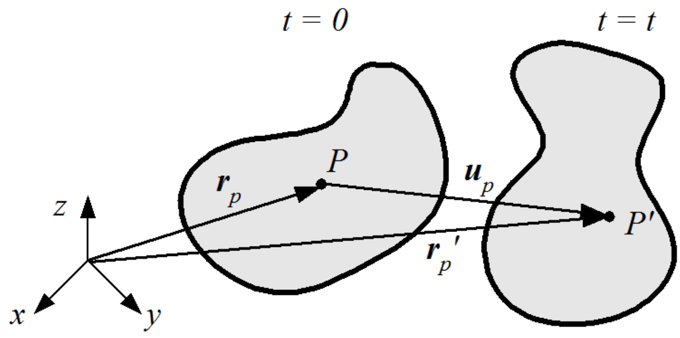


Figure 1.1. Derivation of the displacement vector

Using Cartesian coordinates the $\boldsymbol{r}_P$ and $\boldsymbol{r'}_P$ are the space vectors of P and P', respectively, where $\boldsymbol{r'}_P$ can be described using $\boldsymbol{r}_P$

$$\begin{gathered}\boldsymbol{r}_P = x_P\boldsymbol{i} + y_P\boldsymbol{j} + z_P\boldsymbol{k}, \\ \boldsymbol{r'}_P = \boldsymbol{r}_P + \boldsymbol{u}_P,\end{gathered} \tag{1.1}$$

where $\boldsymbol{u}_P$ is the displacement vector of point P,

$$\boldsymbol{u}_P = u_P\boldsymbol{i} + v_P\boldsymbol{j} + w_P\boldsymbol{k} \tag{1.2}$$

Here u_P, v_P and w_P are the displacement coordinates in the x, y and z directions, respectively. It can be seen that the displacement vector $\boldsymbol{u}$ will vary continuously from point to point, so it forms a displacement field of the body,

$$\boldsymbol{u} = \boldsymbol{u}(\boldsymbol{r}) = u\boldsymbol{i} + v\boldsymbol{j} + w\boldsymbol{k} \tag{1.3}$$

We usually express it as a function of the coordinates of the undeformed geometry,

$$u = u(\boldsymbol{r}) = u(x, y, z,); \quad v = v(\boldsymbol{r}) = v(x, y, z); \quad w = w(\boldsymbol{r}) = w(x, y, z) \tag{1.4}$$

The unit of the displacement is in *mm*.

Derivative tensor and its decomposition

Consider a Q point which is in the very small domain about the point P, and $P \neq Q$. We have formed the vector $\Delta\boldsymbol{r}$ connecting these points by a directed line segment shown in Figure 1.2. The difference of the $\boldsymbol{u}_P$ and $\boldsymbol{u}_Q$ is the relative displacement vector $\Delta\boldsymbol{u}$. An elastic solid is said to be deformed or strained when the relative displacements between points in the body are changed. This is in contrast to rigid body motion, where the distance between points remains the same.

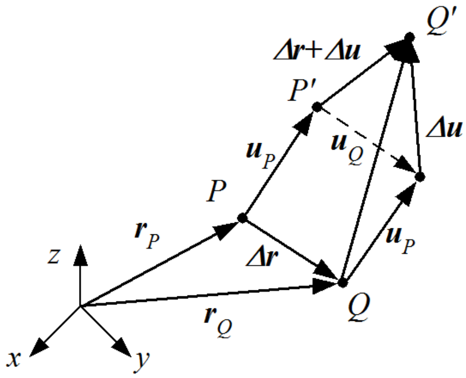


Figure 1.2. General deformation between two neighbouring points

It can be seen that $\Delta\boldsymbol{r}$ can be expressed

$$\begin{aligned}\Delta\boldsymbol{r} = \boldsymbol{r}_Q - \boldsymbol{r}_P = (x_Q - x_P)\boldsymbol{i} + (y_Q - y_P)\boldsymbol{j} + (z_Q - z_P)\boldsymbol{k} = \\ = \Delta x\boldsymbol{i} + \Delta y\boldsymbol{j} + \Delta z\boldsymbol{k}\end{aligned} \tag{1.5}$$

Since P and Q are neighbouring points, we can use a Taylor series expansion around point Q to express the components. Note, that the higher-order terms of the expansion have been dropped since the components of $\boldsymbol{r}$ are small,

$$\boldsymbol{u_Q} \cong \boldsymbol{u_P} + \frac{\partial \boldsymbol{u_P}}{\partial x}\Delta x + \frac{\partial \boldsymbol{u_P}}{\partial y}\Delta y + \frac{\partial \boldsymbol{u_P}}{\partial z}\Delta z,$$
$$\boldsymbol{v_Q} \cong \boldsymbol{v_P} + \frac{\partial \boldsymbol{v_P}}{\partial x}\Delta x + \frac{\partial \boldsymbol{v_P}}{\partial y}\Delta y + \frac{\partial \boldsymbol{v_P}}{\partial z}\Delta z, \quad (1.6)$$
$$\boldsymbol{w_Q} \cong \boldsymbol{w_P} + \frac{\partial \boldsymbol{w_P}}{\partial x}\Delta x + \frac{\partial \boldsymbol{w_P}}{\partial y}\Delta y + \frac{\partial \boldsymbol{w_P}}{\partial z}\Delta z.$$

It can be written in vector form

$$\boldsymbol{u}_Q \cong \boldsymbol{u}_P + \begin{bmatrix}\frac{\partial \boldsymbol{u}_P}{\partial x} & \frac{\partial \boldsymbol{u}_P}{\partial y} & \frac{\partial \boldsymbol{u}_P}{\partial z}\end{bmatrix}\Delta \boldsymbol{r}, \quad (1.7)$$

from where the approximation of the relative displacement vector is

$$\Delta \boldsymbol{u} = \boldsymbol{u}_Q - \boldsymbol{u}_P \cong \begin{bmatrix}\frac{\partial \boldsymbol{u}_P}{\partial x} & \frac{\partial \boldsymbol{u}_P}{\partial y} & \frac{\partial \boldsymbol{u}_P}{\partial z}\end{bmatrix}\Delta \boldsymbol{r}. \quad (1.8)$$

We can introduce now the derivative tensor $\boldsymbol{U}_P$ of the displacement field

$$\Delta \boldsymbol{u} = \boldsymbol{U}_P \Delta \boldsymbol{r}, \quad (1.9)$$

where

$$\boldsymbol{U}_P = \begin{bmatrix}\frac{\partial \boldsymbol{u}_P}{\partial x} & \frac{\partial \boldsymbol{u}_P}{\partial y} & \frac{\partial \boldsymbol{u}_P}{\partial z}\end{bmatrix} = \begin{bmatrix}\frac{\partial u_P}{\partial x} & \frac{\partial u_P}{\partial y} & \frac{\partial u_P}{\partial z} \\ \frac{\partial v_P}{\partial x} & \frac{\partial v_P}{\partial y} & \frac{\partial v_P}{\partial z} \\ \frac{\partial w_P}{\partial x} & \frac{\partial w_P}{\partial y} & \frac{\partial w_P}{\partial z}\end{bmatrix}. \quad (1.10)$$

In general

$$\boldsymbol{U} = \begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{bmatrix}. \quad (1.11)$$

Considering the ∇ Hamilton differential operator

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k}, \quad (1.12)$$

the derivative tensor can be written in dyadic form

$$\boldsymbol{U} = \frac{\partial \boldsymbol{u}}{\partial x} \circ \boldsymbol{i} + \frac{\partial \boldsymbol{u}}{\partial y} \circ \boldsymbol{j} + \frac{\partial \boldsymbol{u}}{\partial z} \circ \boldsymbol{k} = \boldsymbol{u} \circ \nabla. \tag{1.13}$$

All tensors can be decomposed into the sum of a symmetric and an asymmetric tensor, so let us write the derivative tensor of the displacement field $\boldsymbol{U}$ into the following form,

$$\boldsymbol{U} = \frac{1}{2}(\boldsymbol{u} \circ \nabla + \nabla \circ \boldsymbol{u}) + \frac{1}{2}(\boldsymbol{u} \circ \nabla - \nabla \circ \boldsymbol{u}) = \frac{1}{2}(\boldsymbol{U} + \boldsymbol{U}^T) + \frac{1}{2}(\boldsymbol{U} - \boldsymbol{U}^T), \tag{1.14}$$

where is $\boldsymbol{U}^T$ the transpose of the derivative tensor,

$$\boldsymbol{U}^T = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial v}{\partial x} & \frac{\partial w}{\partial x} \\ \frac{\partial u}{\partial y} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial y} \\ \frac{\partial u}{\partial z} & \frac{\partial v}{\partial z} & \frac{\partial w}{\partial z} \end{bmatrix}. \tag{1.15}$$

The first part is a symmetric tensor called strain tensor,

$$\boldsymbol{A} = \boldsymbol{U}_{symmetric} = \frac{1}{2}(\boldsymbol{U} + \boldsymbol{U}^T). \tag{1.16}$$

The second part is an asymmetric tensor called rotation tensor,

$$\boldsymbol{\Psi} = \boldsymbol{U}_{asymmetric} = \frac{1}{2}(\boldsymbol{U} - \boldsymbol{U}^T). \tag{1.17}$$

Using the Eq. 1.16 and Eq. 1.17 the derivative tensor can be written with the sum of the strain tensor and the rotation tensor,

$$\boldsymbol{U} = \boldsymbol{A} + \boldsymbol{\Psi}. \tag{1.18}$$

The strain tensor represents the pure deformation of the element, while the rotation tensor represents the rigid-body rotation of the element. The decomposition is illustrated for a two-dimensional case in Figure 1.3.

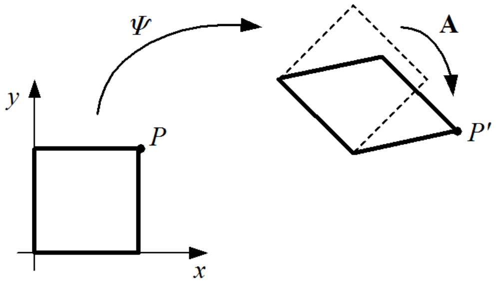


Figure 1.3. The physical interpretation of the strain tensor and the rotation tensor

Rotation tensor

The rotation tensor represents the rigid-body rotation of the element. Let's investigate the asymmetric part of the derivative tensor of the displacement.

$$\boldsymbol{\Psi} = \frac{1}{2}(\boldsymbol{U} - \boldsymbol{U}^T) = \begin{bmatrix} 0 & \frac{1}{2}\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right) \\ \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) & 0 & \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right) \\ \frac{1}{2}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) & 0 \end{bmatrix} = \qquad (1.19)$$

$$= \begin{bmatrix} 0 & -\varphi_z & \varphi_y \\ \varphi_z & 0 & -\varphi_x \\ -\varphi_y & \varphi_x & 0 \end{bmatrix},$$

where φ is the angle displacement and $|\varphi| \ll 1$.

If there is no deformation, when $\boldsymbol{A} = \boldsymbol{0}$, then relative displacement can be expressed by

$$\Delta\boldsymbol{u} = \boldsymbol{\Psi}\Delta\boldsymbol{r}, \qquad (1.20)$$

so the displacement vector of a body point Q can be written

$$\boldsymbol{u}_Q = \boldsymbol{u}_P + \boldsymbol{\Psi}\cdot\Delta\boldsymbol{r} = \boldsymbol{u}_P + \boldsymbol{\varphi}\times\Delta\boldsymbol{r}, \qquad (1.21)$$

where $\boldsymbol{\varphi}$ is the angle displacement vector, $\boldsymbol{\varphi} = \varphi_x\boldsymbol{i} + \varphi_y\boldsymbol{j} + \varphi_z\boldsymbol{k}$ and $\boldsymbol{u}_P$ can be interpreted as a dislocation. The rigid-body rotation gives no strain energy which has an important role in the calculation, so the rigid-body rotation must be constrained during the calculations using constraints.

1.2. Examples for the investigations of the displacement state of solid bodies

Example 1

The displacement field of a solid body is known. The space vector of a body point P is also known.

Data:

$$\boldsymbol{u}(x,y,z) = Axy^2\boldsymbol{i} + Ayz^2\boldsymbol{j} + Azx^2\boldsymbol{k}$$

$$A = 10^{-5}mm^{-2}$$

$$\boldsymbol{r}_P = -40\boldsymbol{i} + 20\boldsymbol{j} + 30\boldsymbol{k}\ (mm)$$

$$\boldsymbol{r}_Q = \boldsymbol{r}_P + \boldsymbol{i} == -39\boldsymbol{i} + 20\boldsymbol{j} + 30\boldsymbol{k}\ (mm)$$

Questions:

a, Calculate the derivative tensor, the strain tensor and the rotation tensor!

b, Calculate the deviation between the exact and the appropriate value of the displacement vector considering $\boldsymbol{r}_Q$.

Solution:

a, To determine the derivative tensor the local coordinate derivatives of the displacement field have to be calculated.

$$\frac{\partial \boldsymbol{u}}{\partial x} = \frac{\partial u}{\partial x}\boldsymbol{i} + \frac{\partial v}{\partial x}\boldsymbol{j} + \frac{\partial w}{\partial x}\boldsymbol{k} = Ay^2\boldsymbol{i} + 2Azx\boldsymbol{k}$$

$$\frac{\partial \boldsymbol{u}}{\partial y} = \frac{\partial u}{\partial y}\boldsymbol{i} + \frac{\partial v}{\partial y}\boldsymbol{j} + \frac{\partial w}{\partial y}\boldsymbol{k} = 2Axy\boldsymbol{i} + Az^2\boldsymbol{j}$$

$$\frac{\partial \boldsymbol{u}}{\partial z} = \frac{\partial u}{\partial z}\boldsymbol{i} + \frac{\partial v}{\partial z}\boldsymbol{j} + \frac{\partial w}{\partial z}\boldsymbol{k} = 2Ayz\boldsymbol{j} + Ax^2\boldsymbol{k}$$

From the local coordinate derivatives of the displacement field the derivative tensor can be established.

$$\boldsymbol{U} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} Ay^2 & 2Axy & 0 \\ 0 & Az^2 & 2Ayz \\ 2Azx & 0 & Ax^2 \end{bmatrix} = A \cdot \begin{bmatrix} y^2 & 2xy & 0 \\ 0 & z^2 & 2yz \\ 2zx & 0 & x^2 \end{bmatrix}$$

After substituting the $\boldsymbol{r}_P$ space vector coordinates the derivative tensor of solid body point P can be determined.

$$\boldsymbol{U}_P = \begin{bmatrix} \frac{\partial u_P}{\partial x} & \frac{\partial u_P}{\partial y} & \frac{\partial u_P}{\partial z} \\ \frac{\partial v_P}{\partial x} & \frac{\partial v_P}{\partial y} & \frac{\partial v_P}{\partial z} \\ \frac{\partial w_P}{\partial x} & \frac{\partial w_P}{\partial y} & \frac{\partial w_P}{\partial z} \end{bmatrix} = A \cdot \begin{bmatrix} y_P^2 & 2x_P y_P & 0 \\ 0 & z_P^2 & 2y_P z_P \\ 2z_P x_P & 0 & x_P^2 \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -16 & 0 \\ 0 & 9 & 12 \\ -24 & 0 & 16 \end{bmatrix} \cdot 10^{-3}$$

Using the decomposition of the derivative tensor, the strain tensor of body point P can be calculated.

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{U} + \boldsymbol{U}^T) = A \cdot \begin{bmatrix} y^2 & xy & zx \\ xy & z^2 & yz \\ zx & yz & x^2 \end{bmatrix}$$

$$\boldsymbol{A}_P = A \cdot \begin{bmatrix} y_P^2 & x_P y_P & z_P x_P \\ x_P y_P & z_P^2 & y_P z_P \\ z_P x_P & y_P z_P & x_P^2 \end{bmatrix} = \begin{bmatrix} 4 & -8 & -12 \\ -8 & 9 & 6 \\ -12 & 6 & 16 \end{bmatrix} \cdot 10^{-3}$$

Using the decomposition of the derivative tensor, the rotation tensor of body point P can be calculated.

$$\boldsymbol{\Psi} = \frac{1}{2}(\boldsymbol{U} - \boldsymbol{U}^T) = A \cdot \begin{bmatrix} 0 & xy & -zx \\ -xy & 0 & yz \\ zx & -yz & 0 \end{bmatrix}$$

$$\boldsymbol{\Psi}_P = A \cdot \begin{bmatrix} 0 & x_P y_P & -z_P x_P \\ -x_P y_P & 0 & y_P z_P \\ z_P x_P & -y_P z_P & 0 \end{bmatrix} = \begin{bmatrix} 0 & -8 & 12 \\ 8 & 0 & 6 \\ -12 & -6 & 0 \end{bmatrix} \cdot 10^{-3}$$

b, After substitution the displacement vector of body point P can be determined.

$$\boldsymbol{u}_P = 10^{-5} \cdot (-40) \cdot 20^2 \boldsymbol{i} + 10^{-5} \cdot 20 \cdot 30^2 \boldsymbol{j} + 10^{-5} \cdot 30 \cdot (-40)^2 \boldsymbol{k}$$

$$\boldsymbol{u}_P = -0.16\boldsymbol{i} + 0.18\boldsymbol{j} + 0.48\boldsymbol{k}\ (mm)$$

After substitution the displacement vector of body point Q can be determined. This will be the exact solution of the displacement vector at point Q.

$$\boldsymbol{u}_Q = 10^{-5} \cdot (-39) \cdot 20^2 \boldsymbol{i} + 10^{-5} \cdot 20 \cdot 30^2 \boldsymbol{j} + 10^{-5} \cdot 30 \cdot (-39)^2 \boldsymbol{k}$$

$$\boldsymbol{u}_Q = -0.156\boldsymbol{i} + 0.18\boldsymbol{j} + 0.4563\boldsymbol{k}\ (mm) = \boldsymbol{u}_{Qexact}$$

SZÉCHENYI 2020

While

$$\boldsymbol{r}_Q = \boldsymbol{r}_P + \boldsymbol{i} == -39\boldsymbol{i} + 20\boldsymbol{j} + 30\boldsymbol{k}\ (mm)$$

the relative space vector is

$$\Delta\boldsymbol{r} = \boldsymbol{r}_Q - \boldsymbol{r}_P = \boldsymbol{i}$$

The approximate solution of the displacement vector at point Q can be determined using Equation 1.8 and Equation 1.9.

$$\boldsymbol{u}_{Qapproximate} \cong \boldsymbol{u}_P + \boldsymbol{U}_P \cdot \Delta\boldsymbol{r} = \begin{bmatrix} -0.16 \\ 0.18 \\ 0.48 \end{bmatrix} + 10^{-3} \cdot \begin{bmatrix} 4 & -16 & 0 \\ 0 & 9 & 12 \\ -24 & 0 & 16 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\boldsymbol{u}_{Qapproximate} = \begin{bmatrix} -0.16 \\ 0.18 \\ 0.48 \end{bmatrix} + \begin{bmatrix} 0.004 \\ 0 \\ -0.024 \end{bmatrix} = \begin{bmatrix} -0.156 \\ 0.18 \\ 0.456 \end{bmatrix} (mm)$$

The relative displacement vector can be calculated.

$$\Delta\boldsymbol{u}_Q = \boldsymbol{u}_{Qexact} - \boldsymbol{u}_{Qapproximate} = \begin{bmatrix} 0 \\ 0 \\ 0.0003 \end{bmatrix} (mm)$$

Example 2

The elements of the derivative tensor are known at body point *P*.

Data:

$$\boldsymbol{U}_P = \begin{bmatrix} 3 & 0 & -4 \\ 10 & -8 & -12 \\ 0 & 0 & 6 \end{bmatrix} \cdot 10^{-4}$$

Questions:

a, Using the decomposition of the derivative tensor determine the strain tensor of body point *P*.

b, Using the decomposition of the derivative tensor determine the rotation tensor of body point *P*.

Solution:

The derivative tensor can be decomposed into a symmetric and an asymmetric tensor.

$$\boldsymbol{U} = \boldsymbol{A} + \boldsymbol{\Psi} = \frac{1}{2}(\boldsymbol{U} + \boldsymbol{U}^T) + \frac{1}{2}(\boldsymbol{U} - \boldsymbol{U}^T)$$

The transpose of the derivative tensor is

$$\boldsymbol{U}_P^T = \begin{bmatrix} 3 & 10 & 0 \\ 0 & -8 & 0 \\ -4 & -12 & 6 \end{bmatrix} \cdot 10^{-4}$$

The strain tensor of point P

$$\boldsymbol{A}_P = \frac{1}{2}(\boldsymbol{U}_P + \boldsymbol{U}_P^T) = \begin{bmatrix} 3 & 5 & -2 \\ 5 & -8 & -6 \\ -2 & -6 & 6 \end{bmatrix} \cdot 10^{-4}$$

The rotation tensor of point P

$$\boldsymbol{\Psi}_P = \frac{1}{2}(\boldsymbol{U}_P - \boldsymbol{U}_P^T) = \begin{bmatrix} 0 & -5 & -2 \\ 5 & 0 & -6 \\ 2 & 6 & 0 \end{bmatrix} \cdot 10^{-4}$$

Example 3

The displacement field of a solid body is known. The space vector of a body point P is also known.

Data:

$$\boldsymbol{u}(x,y,z) = Ax^3y\boldsymbol{i} + Ay^3z\boldsymbol{j} + Az^3x\boldsymbol{k}$$

$$A = 10^{-4} mm^{-3}$$

$$\boldsymbol{r}_P = 2\boldsymbol{i} - 3\boldsymbol{j} + \boldsymbol{k}\ (mm)$$

$$\boldsymbol{r}_Q = \boldsymbol{r}_P + \boldsymbol{j} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}\ (mm)$$

Questions:

a, Calculate the derivative tensor, the strain tensor and the rotation tensor!

b, Calculate the deviation between the exact and the appropriate value of the displacement vector considering $\boldsymbol{r}_Q$.

Solution:

a, To determine the derivative tensor the local coordinate derivatives of the displacement field have to be calculated.

$$\frac{\partial \boldsymbol{u}}{\partial x} = \frac{\partial u}{\partial x}\boldsymbol{i} + \frac{\partial v}{\partial x}\boldsymbol{j} + \frac{\partial w}{\partial x}\boldsymbol{k} = 3Ax^2y\boldsymbol{i} + Az^3\boldsymbol{k}$$

$$\frac{\partial \boldsymbol{u}}{\partial y} = \frac{\partial u}{\partial y}\boldsymbol{i} + \frac{\partial v}{\partial y}\boldsymbol{j} + \frac{\partial w}{\partial y}\boldsymbol{k} = Ax^3\boldsymbol{i} + 3Ay^2z\boldsymbol{j}$$

$$\frac{\partial \boldsymbol{u}}{\partial z} = \frac{\partial u}{\partial z}\boldsymbol{i} + \frac{\partial v}{\partial z}\boldsymbol{j} + \frac{\partial w}{\partial z}\boldsymbol{k} = Ay^3\boldsymbol{j} + 3Az^2x\boldsymbol{k}$$

From the local coordinate derivatives of the displacement field the derivative tensor can be established.

$$\boldsymbol{U}=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z}\\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z}\\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z}\end{bmatrix}=\begin{bmatrix}3Ax^2y & Ax^3 & 0\\ 0 & 3Ay^2z & Ay^3\\ Az^3 & 0 & 3Az^2x\end{bmatrix}=A\cdot\begin{bmatrix}3x^2y & x^3 & 0\\ 0 & 3y^2z & y^3\\ z^3 & 0 & 3z^2x\end{bmatrix}$$

After substituting the $\boldsymbol{r}_P$ space vector coordinates the derivative tensor of solid body point P can be determined.

$$\boldsymbol{U}_P=\begin{bmatrix}\frac{\partial u_P}{\partial x} & \frac{\partial u_P}{\partial y} & \frac{\partial u_P}{\partial z}\\ \frac{\partial v_P}{\partial x} & \frac{\partial v_P}{\partial y} & \frac{\partial v_P}{\partial z}\\ \frac{\partial w_P}{\partial x} & \frac{\partial w_P}{\partial y} & \frac{\partial w_P}{\partial z}\end{bmatrix}=A\cdot\begin{bmatrix}3x_P^2y_P & x_P^3 & 0\\ 0 & 3y_P^2z_P & y_P^3\\ z_P^3 & 0 & 3z_P^2x_P\end{bmatrix}$$

$$=\begin{bmatrix}-36 & 8 & 0\\ 0 & 27 & -27\\ 1 & 0 & 6\end{bmatrix}\cdot 10^{-4}$$

Using the decomposition of the derivative tensor, the strain tensor of body point P can be calculated.

$$\boldsymbol{A}=\frac{1}{2}(\boldsymbol{U}+\boldsymbol{U}^T)=\frac{1}{2}\left(A\cdot\begin{bmatrix}3x^2y & x^3 & 0\\ 0 & 3y^2z & y^3\\ z^3 & 0 & 3z^2x\end{bmatrix}+A\cdot\begin{bmatrix}3x^2y & 0 & z^3\\ x^3 & 3y^2z & 0\\ 0 & y^3 & 3z^2x\end{bmatrix}\right)=$$

$$=A\cdot\begin{bmatrix}3x^2y & 0.5x^3 & 0.5z^3\\ 0.5x^3 & 3y^2z & 0.5y^3\\ 0.5z^3 & 0.5y^3 & 3z^2x\end{bmatrix}$$

$$\boldsymbol{A}_P=A\cdot\begin{bmatrix}3x_P^2y_P & 0.5x_P^3 & 0.5z_P^3\\ 0.5x_P^3 & 3y_P^2z_P & 0.5y_P^3\\ 0.5z_P^3 & 0.5y_P^3 & 3z_P^2x_P\end{bmatrix}=\begin{bmatrix}-36 & 4 & 0.5\\ 4 & 27 & -13.5\\ 0.5 & -13.5 & 6\end{bmatrix}\cdot 10^{-4}$$

Using the decomposition of the derivative tensor, the rotation tensor of body point P can be calculated.

$$\boldsymbol{\Psi}=\frac{1}{2}(\boldsymbol{U}-\boldsymbol{U}^T)=\frac{1}{2}\left(A\cdot\begin{bmatrix}3x^2y & x^3 & 0\\ 0 & 3y^2z & y^3\\ z^3 & 0 & 3z^2x\end{bmatrix}-A\cdot\begin{bmatrix}3x^2y & 0 & z^3\\ x^3 & 3y^2z & 0\\ 0 & y^3 & 3z^2x\end{bmatrix}\right)=$$

$$= A \cdot \begin{bmatrix} 0 & 0.5x^3 & -0.5z^3 \\ -0.5x^3 & 0 & 0.5y^3 \\ 0.5z^3 & -0.5y^3 & 0 \end{bmatrix}$$

$$\boldsymbol{\Psi}_P = A \cdot \begin{bmatrix} 0 & 0.5x_P^3 & -0.5z_P^3 \\ -0.5\ \ x_P^3 & 0 & 0.5y_P^3 \\ 0.5z_P^3 & -0.5y_P^3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 4 & -0.5 \\ -4 & 0 & -13.5 \\ 0.5 & 13.5 & 0 \end{bmatrix} \cdot 10^{-3}$$

b, After substitution the displacement vector of body point P can be determined.

$$\boldsymbol{u}_P = 10^{-4} \cdot 2^3 \cdot (-3)\boldsymbol{i} + 10^{-4} \cdot (-3)^3 \cdot 1\boldsymbol{j} + 10^{-4} \cdot 1^3 \cdot 2\boldsymbol{k}$$

$$\boldsymbol{u}_P = -0.0024\boldsymbol{i} - 0.0027\boldsymbol{j} + 0.0002\boldsymbol{k}\ (mm)$$

After substitution the displacement vector of body point Q can be determined. This will be the exact solution of the displacement vector at point Q.

$$\boldsymbol{u}_Q = 10^{-4} \cdot 2^3 \cdot (-2)\boldsymbol{i} + 10^{-4} \cdot (-2)^3 \cdot 1\boldsymbol{j} + 10^{-4} \cdot 1^3 \cdot 2\boldsymbol{k}$$

$$\boldsymbol{u}_Q = -0.0016\boldsymbol{i} - 0.0008\boldsymbol{j} + 0.0002\boldsymbol{k}\ (mm) = \boldsymbol{u}_{Qexact}$$

While

$$\boldsymbol{r}_Q = \boldsymbol{r}_P + \boldsymbol{j} = 2\boldsymbol{i} - 2\boldsymbol{j} + \boldsymbol{k}\ (mm)$$

the relative space vector is

$$\Delta\boldsymbol{r} = \boldsymbol{r}_Q - \boldsymbol{r}_P = \boldsymbol{j}$$

The approximate solution of the displacement vector at point Q can be determined using Equation 1.8 and Equation 1.9.

$$\boldsymbol{u}_{Qapproximate} \cong \boldsymbol{u}_P + \boldsymbol{U}_P \cdot \Delta\boldsymbol{r} = \begin{bmatrix} -0.0024 \\ -0.0027 \\ 0.0002 \end{bmatrix} + 10^{-4} \cdot \begin{bmatrix} -36 & 8 & 0 \\ 0 & 27 & -27 \\ 1 & 0 & 6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\boldsymbol{u}_{Qapproximate} = \begin{bmatrix} -0.0024 \\ -0.0027 \\ 0.0002 \end{bmatrix} + \begin{bmatrix} 0.0008 \\ 0.0027 \\ 0 \end{bmatrix} = \begin{bmatrix} -0.0016 \\ 0 \\ 0.0002 \end{bmatrix} (mm)$$

The relative displacement vector can be calculated.

$$\Delta\boldsymbol{u}_Q = \boldsymbol{u}_{Qexact} - \boldsymbol{u}_{Qapproximate} = \begin{bmatrix} 0 \\ -0.0008 \\ 0 \end{bmatrix} (mm)$$

Example 4

The displacement field of a solid body is known. The space vector of a body point P is also known.

Data:

$$\boldsymbol{u}(x,y,z) = Ay^2\boldsymbol{i} + Az^2\boldsymbol{j} + Ax^2\boldsymbol{k}$$

$$A = 10^{-3} mm^{-1}$$

$$\boldsymbol{r}_P = -10\boldsymbol{i} + 8\boldsymbol{j} + 6\boldsymbol{k}\ (mm)$$

$$\boldsymbol{r}_Q = \boldsymbol{r}_P + \boldsymbol{k} = -10\boldsymbol{i} + 8\boldsymbol{j} + 7\boldsymbol{k}\ (mm)$$

Questions:

a, Calculate the derivative tensor, the strain tensor and the rotation tensor!

b, Calculate the deviation between the exact and the appropriate value of the displacement vector considering $\boldsymbol{r}_Q$.

Solution:

a, To determine the derivative tensor the local coordinate derivatives of the displacement field have to be calculated.

$$\frac{\partial \boldsymbol{u}}{\partial x} = \frac{\partial u}{\partial x}\boldsymbol{i} + \frac{\partial v}{\partial x}\boldsymbol{j} + \frac{\partial w}{\partial x}\boldsymbol{k} = 2Ax\boldsymbol{k}$$

$$\frac{\partial \boldsymbol{u}}{\partial y} = \frac{\partial u}{\partial y}\boldsymbol{i} + \frac{\partial v}{\partial y}\boldsymbol{j} + \frac{\partial w}{\partial y}\boldsymbol{k} = 2Ay\boldsymbol{i}$$

$$\frac{\partial \boldsymbol{u}}{\partial z} = \frac{\partial u}{\partial z}\boldsymbol{i} + \frac{\partial v}{\partial z}\boldsymbol{j} + \frac{\partial w}{\partial z}\boldsymbol{k} = 2Az\boldsymbol{j}$$

From the local coordinate derivatives of the displacement field the derivative tensor can be established.

$$\boldsymbol{U} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{bmatrix} = \begin{bmatrix} 0 & 2Ay & 0 \\ 0 & 0 & 2Az \\ 2Ax & 0 & 0 \end{bmatrix} = A \cdot \begin{bmatrix} 0 & 2y & 0 \\ 0 & 0 & 2z \\ 2x & 0 & 0 \end{bmatrix}$$

After substituting the $\boldsymbol{r}_P$ space vector coordinates the derivative tensor of solid body point P can be determined.

$$\boldsymbol{U}_P = \begin{bmatrix} \frac{\partial u_P}{\partial x} & \frac{\partial u_P}{\partial y} & \frac{\partial u_P}{\partial z} \\ \frac{\partial v_P}{\partial x} & \frac{\partial v_P}{\partial y} & \frac{\partial v_P}{\partial z} \\ \frac{\partial w_P}{\partial x} & \frac{\partial w_P}{\partial y} & \frac{\partial w_P}{\partial z} \end{bmatrix} = A \cdot \begin{bmatrix} 0 & 2y_P & 0 \\ 0 & 0 & 2z_P \\ 2x_P & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 16 & 0 \\ 0 & 0 & 12 \\ -20 & 0 & 0 \end{bmatrix} \cdot 10^{-3}$$

Using the decomposition of the derivative tensor, the strain tensor of body point P can be calculated.

$$\boldsymbol{A} = \frac{1}{2}(\boldsymbol{U} + \boldsymbol{U}^T) = \frac{1}{2}\left(A \cdot \begin{bmatrix} 0 & 2y & 0 \\ 0 & 0 & 2z \\ 2x & 0 & 0 \end{bmatrix} + A \cdot \begin{bmatrix} 0 & 0 & 2x \\ 2y & 0 & 0 \\ 0 & 2z & 0 \end{bmatrix} \right) =$$

$$= A \cdot \begin{bmatrix} 0 & y & x \\ y & 0 & z \\ x & z & 0 \end{bmatrix}$$

$$\boldsymbol{A}_P = A \cdot \begin{bmatrix} 0 & y_P & x_P \\ y_P & 0 & z_P \\ x_P & z_P & 0 \end{bmatrix} = \begin{bmatrix} 0 & 8 & -10 \\ 8 & 0 & 6 \\ -10 & 6 & 0 \end{bmatrix} \cdot 10^{-3}$$

Using the decomposition of the derivative tensor, the rotation tensor of body point P can be calculated.

$$\boldsymbol{\Psi} = \frac{1}{2}(\boldsymbol{U} - \boldsymbol{U}^T) = \frac{1}{2}\left(A \cdot \begin{bmatrix} 0 & 2y & 0 \\ 0 & 0 & 2z \\ 2x & 0 & 0 \end{bmatrix} - A \cdot \begin{bmatrix} 0 & 0 & 2x \\ 2y & 0 & 0 \\ 0 & 2z & 0 \end{bmatrix} \right) =$$

$$= A \cdot \begin{bmatrix} 0 & y & -x \\ -y & 0 & z \\ x & -z & 0 \end{bmatrix}$$

$$\boldsymbol{\Psi}_P = A \cdot \begin{bmatrix} 0 & y_P & -x_P \\ -y_P & 0 & z_P \\ x_P & -z_P & 0 \end{bmatrix} = \begin{bmatrix} 0 & 8 & -10 \\ -8 & 0 & 6 \\ 10 & -6 & 0 \end{bmatrix} \cdot 10^{-3}$$

b, After substitution the displacement vector of body point P can be determined.

$$\boldsymbol{u}_P = 10^{-3} \cdot 8^2 \boldsymbol{i} + 10^{-3} \cdot 6^2 \boldsymbol{j} + 10^{-3} \cdot (-10)^2 \boldsymbol{k}$$

$$\boldsymbol{u}_P = 0.064\boldsymbol{i} + 0.036\boldsymbol{j} - 0.1\boldsymbol{k}\ (mm)$$

After substitution the displacement vector of body point Q can be determined. This will be the exact solution of the displacement vector at point Q.

$$\boldsymbol{u}_Q = 10^{-3} \cdot 8^2 \boldsymbol{i} + 10^{-3} \cdot 7^2 \boldsymbol{j} + 10^{-3} \cdot (-10)^2 \boldsymbol{k}$$

$$\boldsymbol{u}_Q = 0.064\boldsymbol{i} + 0.049\boldsymbol{j} - 0.1\boldsymbol{k}\ (mm) = \boldsymbol{u}_{Qexact}$$

While

$$\boldsymbol{r}_Q = \boldsymbol{r}_P + \boldsymbol{k} = -10\boldsymbol{i} + 8\boldsymbol{j} + 7\boldsymbol{k}\ (mm)$$

the relative space vector is

$$\Delta\boldsymbol{r} = \boldsymbol{r}_Q - \boldsymbol{r}_P = \boldsymbol{k}$$

The approximate solution of the displacement vector at point Q can be determined using Equation 1.8 and Equation 1.9.

$$\boldsymbol{u}_{Qapproximate} \cong \boldsymbol{u}_P + \boldsymbol{U}_P \cdot \Delta\boldsymbol{r} = \begin{bmatrix} 0.064 \\ 0.036 \\ -0.1 \end{bmatrix} + 10^{-3} \cdot \begin{bmatrix} 0 & 16 & 0 \\ 0 & 0 & 12 \\ -20 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boldsymbol{u}_{Qapproximate} = \begin{bmatrix} 0.064 \\ 0.036 \\ -0.1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.012 \\ 0 \end{bmatrix} = \begin{bmatrix} 0.064 \\ 0.048 \\ -0.1 \end{bmatrix} (mm)$$

The relative displacement vector can be calculated.

$$\Delta\boldsymbol{u}_Q = \boldsymbol{u}_{Qexact} - \boldsymbol{u}_{Qapproximate} = \begin{bmatrix} 0 \\ 0.0001 \\ 0 \end{bmatrix} (mm)$$

SZÉCHENYI 2020

2. STRAIN STATE OF SOLID BODIES

2.1. Theoretical background of the strain state of solid bodies

Strain tensor (state of strain)

The strain tensor represents the pure deformation of the element. Let's investigate the symmetric part of the derivative tensor of the displacement.

$$\boldsymbol{A}=\frac{1}{2}(\boldsymbol{U}+\boldsymbol{U}^T)=\begin{bmatrix}\frac{\partial u}{\partial x} & \frac{1}{2}\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right) & \frac{1}{2}\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)\\ \frac{1}{2}\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right) & \frac{\partial v}{\partial y} & \frac{1}{2}\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)\\ \frac{1}{2}\left(\frac{\partial w}{\partial x}+\frac{\partial u}{\partial z}\right) & \frac{1}{2}\left(\frac{\partial w}{\partial y}+\frac{\partial v}{\partial z}\right) & \frac{\partial w}{\partial z}\end{bmatrix}= \tag{2.1}$$
$$=\begin{bmatrix}\varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz}\\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz}\\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z\end{bmatrix}=\boldsymbol{A}^T,$$

where ε_x, ε_y and ε_z are the normal strains in the x, y and z directions, respectively, and $\varepsilon \ll 1$, $\gamma_{xy}=\gamma_{yx}$, $\gamma_{yz}=\gamma_{zy}$ and $\gamma_{zx}=\gamma_{xz}$ are the so called shear angles and $\gamma \ll 1$ considering small deformations. The normal strain has no unit, while the shear angle's unit is radian. If the $\varepsilon > 0$ the unit length elongates, if the $\varepsilon < 0$ the unit length shortens. If the $\gamma > 0$ the original 90° decreases, if the $\gamma < 0$ the original 90° increases. The deformation of an elementary point is demonstrated in Figure 2.4. We can also introduce the strain vectors $\boldsymbol{\alpha}_x$, $\boldsymbol{\alpha}_y$ and $\boldsymbol{\alpha}_z$ for the strain description,

$$\boldsymbol{A}=\begin{bmatrix}\varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz}\\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz}\\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z\end{bmatrix}=[\boldsymbol{\alpha}_x \quad \boldsymbol{\alpha}_y \quad \boldsymbol{\alpha}_z], \tag{2.2}$$

so

$$\begin{aligned}\boldsymbol{\alpha}_x &= \varepsilon_x \boldsymbol{i} + \frac{1}{2}\gamma_{yx}\boldsymbol{j} + \frac{1}{2}\gamma_{zx}\boldsymbol{k},\\ \boldsymbol{\alpha}_y &= \frac{1}{2}\gamma_{xy}\boldsymbol{i} + \varepsilon_y \boldsymbol{j} + \frac{1}{2}\gamma_{zy}\boldsymbol{k},\\ \boldsymbol{\alpha}_z &= \frac{1}{2}\gamma_{xz}\boldsymbol{i} + \frac{1}{2}\gamma_{yz}\boldsymbol{j} + \varepsilon_z \boldsymbol{k}.\end{aligned} \tag{2.3}$$

The strain tensor can be written in dyadic form using strain vectors

$$\boldsymbol{A} = \boldsymbol{\alpha}_x \circ \boldsymbol{i} + \boldsymbol{\alpha}_y \circ \boldsymbol{j} + \boldsymbol{\alpha}_z \circ \boldsymbol{k}. \tag{2.4}$$

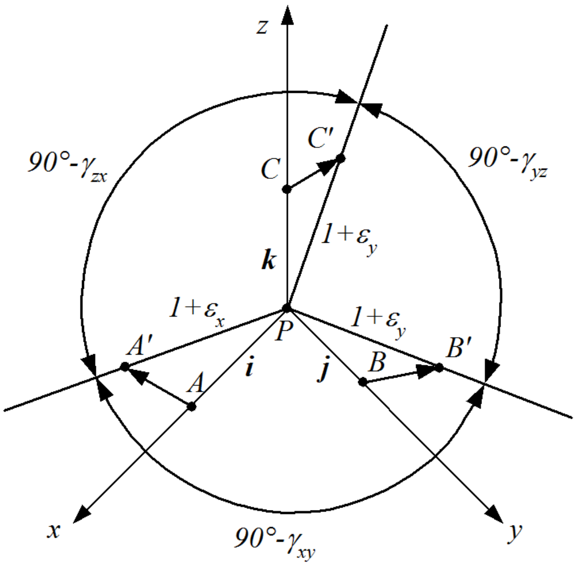


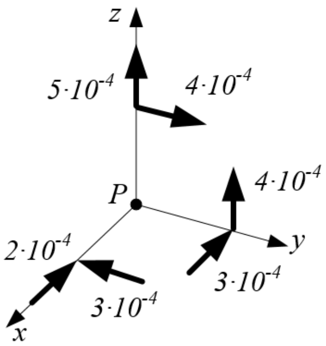
Figure 2.1. Deformation of an elementary point

In Figure 2.1 for the geometrical illustration of the strain state of a body point P we have to order a so called small cube denoted by the $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ unit vectors in the very small domain of P with end points A, B and C.

2.2. Examples for the investigations of the strain state of solid bodies

Example 1

Determination of strain measures of solid body at a given body point! The strain state of the body point P of a solid body is given with the so called elementary cube.



The task is to determine different strain measures from the given data!

Questions:

a, Establish the $\boldsymbol{A}_P$ strain tensor of body point *P*!

b, Determine the $\boldsymbol{\alpha}_x$, $\boldsymbol{\alpha}_y$ and $\boldsymbol{\alpha}_z$ strain vectors which belong to the directions $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$!

c, Determine the following strain measures: ε_x normal strain, γ_{xz} and , γ_{yz} shear angles!

Solution:

a, From the elementary cube the strain measures (normal strains, shear angles) can be read.

The strain tensor in general and after substitution:

$$\boldsymbol{A}_P = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} -2 & -3 & 0 \\ -3 & 0 & 4 \\ 0 & 4 & 5 \end{bmatrix} \cdot 10^{-4}$$

b, The diadic form of the strain tensor:

$$\boldsymbol{A}_P = \boldsymbol{\alpha}_x \circ \boldsymbol{i} + \boldsymbol{\alpha}_y \circ \boldsymbol{j} + \boldsymbol{\alpha}_z \circ \boldsymbol{k}$$

so the strain vector in the *x,y* and *z* directions, respectively:

$$\boldsymbol{\alpha}_x = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{yx} & \frac{1}{2}\gamma_{zx} \end{bmatrix} = [-2 \quad -3 \quad 0] \cdot 10^{-4}$$

$$\boldsymbol{\alpha}_y = \begin{bmatrix} \frac{1}{2}\gamma_{xy} & \varepsilon_y & \frac{1}{2}\gamma_{zy} \end{bmatrix} = [-3 \quad 0 \quad 4] \cdot 10^{-4}$$

$$\boldsymbol{\alpha}_z = \begin{bmatrix} \frac{1}{2}\gamma_{xz} & \frac{1}{2}\gamma_{yz} & \varepsilon_z \end{bmatrix} = [0 \quad 4 \quad 5] \cdot 10^{-4}$$

c, The different strain measures can be read out of the strain tensor!

The normal strain:

$$\varepsilon_x = -2 \cdot 10^{-4}$$

The shear angles:

$$\gamma_{xz} = 0$$

$$\gamma_{yz} = 8 \cdot 10^{-4} rad = 8 \cdot 10^{-4} \frac{180°}{\pi} = 0.04584°$$

Example 2

Calculations with strain measures! The strain measures of a body point P is given and the $\boldsymbol{n}$ and $\boldsymbol{m}$ unit vectors as well.

Data:

$$\varepsilon_x = -10 \cdot 10^{-5}, \varepsilon_y = 8 \cdot 10^{-5}, \varepsilon_z = 5 \cdot 10^{-5}, \gamma_{xz} = \gamma_{xy} = 0, \gamma_{yz} = -5 \cdot 10^{-5}$$

$$\boldsymbol{n} = -0.6\boldsymbol{i} + 0.8\boldsymbol{k}, \quad \boldsymbol{m} = 0.8\boldsymbol{i} + 0.6\boldsymbol{k}$$

Questions:

a, Establish the $\mathbf{A}_P$ strain tensor of body point P!

b, Determine the ε_n normal strain in direction $\boldsymbol{n}$! Determine the γ_{mn} shear angle!

c, Determine the ε_m normal strain in direction $\boldsymbol{m}$! Determine the γ_{nm} shear angle!

d, Determine the γ_{ym} shear angle!

Solution:

a, The strain tensor in general and after substitution:

$$\boldsymbol{A}_P = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} -10 & 0 & 0 \\ 0 & 8 & -2.5 \\ 0 & -2.5 & 2 \end{bmatrix} 10^{-5}$$

b,

$$\varepsilon_n = \boldsymbol{n} \cdot \boldsymbol{A}_P \cdot \boldsymbol{n} = \boldsymbol{\alpha}_n \cdot \boldsymbol{n} = [6 \quad -2 \quad 1.6] \begin{bmatrix} -0.6 \\ 0 \\ 0.8 \end{bmatrix} 10^{-5} = -2.32 \cdot 10^{-5}$$

$$\boldsymbol{\alpha}_n = \boldsymbol{A}_P \cdot \boldsymbol{n} = 10^{-5} \begin{bmatrix} -10 & 0 & 0 \\ 0 & 8 & -2.5 \\ 0 & -2.5 & 2 \end{bmatrix} \begin{bmatrix} -0.6 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 1.6 \end{bmatrix} 10^{-5}$$

$$\gamma_{mn} = 2 \cdot \boldsymbol{m} \cdot \boldsymbol{A}_P \cdot \boldsymbol{n} = 2 \cdot \boldsymbol{\alpha}_n \cdot \boldsymbol{m} = 2 \cdot [6 \quad -2 \quad 1.6] \begin{bmatrix} 0.8 \\ 0 \\ 0.6 \end{bmatrix} 10^{-5} = 11.52 \cdot 10^{-5}$$

c,

$$\varepsilon_m = \boldsymbol{m} \cdot \boldsymbol{A}_P \cdot \boldsymbol{m} = \boldsymbol{\alpha}_m \cdot \boldsymbol{m} = [-8 \quad -1.5 \quad 1.2] \begin{bmatrix} 0.8 \\ 0 \\ 0.6 \end{bmatrix} 10^{-5} = -5.68 \cdot 10^{-5}$$

$$\boldsymbol{\alpha}_m = \boldsymbol{A}_P \cdot \boldsymbol{m} = 10^{-5} \begin{bmatrix} -10 & 0 & 0 \\ 0 & 8 & -2.5 \\ 0 & -2.5 & 2 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0 \\ 0.6 \end{bmatrix} = \begin{bmatrix} -8 \\ -1.5 \\ 1.2 \end{bmatrix} 10^{-5}$$

$$\gamma_{nm} = 2 \cdot \boldsymbol{n} \cdot \boldsymbol{A}_P \cdot \boldsymbol{m} = 2 \cdot \boldsymbol{\alpha}_m \cdot \boldsymbol{n} = 2 \cdot [-8 \quad -1.5 \quad 1.2] \begin{bmatrix} -0.6 \\ 0 \\ 0.8 \end{bmatrix} 10^{-5}$$
$$= 11.52 \cdot 10^{-5} = \gamma_{mn}$$

d,

$$\gamma_{ym} = 2 \cdot \boldsymbol{j} \cdot \boldsymbol{A}_p \cdot \boldsymbol{m} = 2 \cdot \boldsymbol{\alpha}_m \cdot \boldsymbol{j} = 2 \cdot [-8 \quad -1.5 \quad 1.2] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} 10^{-5} = -1.5 \cdot 10^{-5}$$

Example 3

Calculations with strain measures!

The strain measures of a body point P is given and the $\boldsymbol{n}$ and $\boldsymbol{m}$ unit vectors as well.

Data:

$$\varepsilon_x = 5 \cdot 10^{-3}, \varepsilon_y = 4 \cdot 10^{-3}, \varepsilon_z = 10 \cdot 10^{-3}, \gamma_{zx} = -10 \cdot 10^{-3}, \gamma_{xy} = \gamma_{yz} = 0$$

$$\boldsymbol{n} = 0.8\boldsymbol{i} + 0.6\boldsymbol{k}, \boldsymbol{m} = -0.6\boldsymbol{i} + 0.8\boldsymbol{k}$$

Questions:

a, Establish the $\boldsymbol{A}_P$ strain tensor of body point P!

b, Determine the ε_n normal strain in direction $\boldsymbol{n}$! Determine the γ_{mn} shear angle!

c, Determine the ε_m normal strain in direction $\boldsymbol{m}$! Determine the γ_{nm} shear angle!

d, Determine the γ_{ny} shear angle!

Solution:

a, The strain tensor in general and after substitution:

$$\boldsymbol{A}_P = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} 5 & 0 & -5 \\ 0 & 4 & 0 \\ -5 & 0 & 10 \end{bmatrix} 10^{-3}$$

b,

$$\varepsilon_n = \boldsymbol{n} \cdot \boldsymbol{A}_P \cdot \boldsymbol{n} = \boldsymbol{\alpha}_n \cdot \boldsymbol{n}$$

The $\boldsymbol{\alpha}_n$ strain vector can be calculated.

$$\boldsymbol{\alpha}_n = \boldsymbol{A}_P \cdot \boldsymbol{n} = \begin{bmatrix} 5 & 0 & -5 \\ 0 & 4 & 0 \\ -5 & 0 & 10 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0 \\ 0.6 \end{bmatrix} \cdot 10^{-3} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cdot 10^{-3}$$

With the usage of $\boldsymbol{\alpha}_n$ strain vector the ε_n normal strain and the γ_{mn} shear angle can be calculated.

$$\varepsilon_n = \boldsymbol{\alpha}_n \cdot \boldsymbol{n} = [1 \quad 0 \quad 2] \begin{bmatrix} 0.8 \\ 0 \\ 0.6 \end{bmatrix} \cdot 10^{-3} = (0.8 + 1.2) \cdot 10^{-3} = 2 \cdot 10^{-3}$$

$$\gamma_{mn} = 2\boldsymbol{\alpha}_n \cdot \boldsymbol{m} = 2 \cdot [1 \quad 0 \quad 2] \begin{bmatrix} -0.6 \\ 0 \\ 0.8 \end{bmatrix} \cdot 10^{-3} = (-1.2 + 3.2) \cdot 10^{-3} = 2 \cdot 10^{-3}$$

c,

The $\boldsymbol{\alpha}_m$ strain vector can be calculated.

$$\boldsymbol{\alpha}_m = \boldsymbol{A}_P \cdot \boldsymbol{m} = \begin{bmatrix} 5 & 0 & -5 \\ 0 & 4 & 0 \\ -5 & 0 & 10 \end{bmatrix} \begin{bmatrix} -0.6 \\ 0 \\ 0.8 \end{bmatrix} \cdot 10^{-3} = \begin{bmatrix} -7 \\ 0 \\ 11 \end{bmatrix} \cdot 10^{-3}$$

With the usage of $\boldsymbol{\alpha}_m$ strain vector the ε_m normal strain and the γ_{nm} shear angle can be calculated.

$$\varepsilon_m = \boldsymbol{\alpha}_m \cdot \boldsymbol{m} = [-7 \quad 0 \quad 11] \begin{bmatrix} -0.6 \\ 0 \\ 0.8 \end{bmatrix} \cdot 10^{-3} = (4.2 + 8.8) \cdot 10^{-3} = 13 \cdot 10^{-3}$$

$$\gamma_{nm} = 2 \cdot \boldsymbol{\alpha}_m \cdot \boldsymbol{n} = 2 \cdot [-7 \quad 0 \quad 11] \begin{bmatrix} 0.8 \\ 0 \\ 0.6 \end{bmatrix} \cdot 10^{-3} = (-11.2 + 13.2) \cdot 10^{-3}$$
$$= 2 \cdot 10^{-3}$$

d,

$$\gamma_{ny} = 2 \cdot \boldsymbol{\alpha}_n \cdot \boldsymbol{j} = 2 \cdot [1 \quad 0 \quad 2] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot 10^{-3} = 0$$

Example 4

The $\boldsymbol{\alpha}_x$, $\boldsymbol{\alpha}_y$ and $\boldsymbol{\alpha}_z$ strain vectors are given of a solid body point P in the x,y and z directions, respectively. The unit vectors $\boldsymbol{m}$ and $\boldsymbol{n}$ denote two directions and are perpendicular to each other.

Data:

$$\boldsymbol{\alpha}_x = (3\boldsymbol{i} - 4\boldsymbol{k}) \cdot 10^{-4}$$

$$\boldsymbol{\alpha}_y = (-2\boldsymbol{j} + 5\boldsymbol{k}) \cdot 10^{-4}$$

$$\boldsymbol{\alpha}_z = (-4\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k}) \cdot 10^{-4}$$

$$\boldsymbol{m} = -0.8\boldsymbol{j} - 0.6\boldsymbol{k}, \boldsymbol{n} = 0.6\boldsymbol{j} - 0.8\boldsymbol{k}$$

Questions:

a, Using the strain vectors establish the $\boldsymbol{A}_P$ strain tensor!

b, Determine the $\boldsymbol{\alpha}_n$ strain vector in $\boldsymbol{n}$ direction and the ε_n normal strain and the γ_{mn} shear angle!

c, Determine the $\boldsymbol{\alpha}_m$ strain vector in $\boldsymbol{m}$ direction and the ε_m normal strain and the γ_{nm} shear angle!

d, Determine the γ_{mx} and γ_{xy} shear angles on the given planes!

Solution:

a, The strain tensor in general and after substitution:

$$\boldsymbol{A}_P = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix} = \begin{bmatrix} 3 & 0 & -4 \\ 0 & -2 & 5 \\ -4 & 5 & 1 \end{bmatrix} 10^{-4}$$

b,

$$\varepsilon_n = \boldsymbol{n} \cdot \boldsymbol{A}_P \cdot \boldsymbol{n} = \boldsymbol{\alpha}_n \cdot \boldsymbol{n}$$

The $\boldsymbol{\alpha}_n$ strain vector can be calculated.

$$\boldsymbol{\alpha}_n = \boldsymbol{A}_P \cdot \boldsymbol{n} = \begin{bmatrix} 3 & 0 & -4 \\ 0 & -2 & 5 \\ -4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.6 \\ -0.8 \end{bmatrix} \cdot 10^{-4} =$$

$$= \begin{bmatrix} 3\cdot 0 + 0\cdot 0.6 + (-4)\cdot(-0.8) \\ 0\cdot 0 + (-2)\cdot 0.6 + 5\cdot(-0.8) \\ (-4)\cdot 0 + 5\cdot 0.6 + 1\cdot(-0.8) \end{bmatrix} \cdot 10^{-4} = \begin{bmatrix} 3.2 \\ -5.2 \\ 2.2 \end{bmatrix} \cdot 10^{-4}$$

With the usage of $\boldsymbol{\alpha}_n$ strain vector the ε_n normal strain and the γ_{mn} shear angle can be calculated.

$$\varepsilon_n = \boldsymbol{\alpha}_n \cdot \boldsymbol{n} = [3.2 \quad -5.2 \quad 2.2] \begin{bmatrix} 0 \\ 0.6 \\ -0.8 \end{bmatrix} \cdot 10^{-4} =$$

$$= [3.2\cdot 0 + (-5.2)\cdot 0.6 + 2.2\cdot(-0.8)] \cdot 10^{-4} = -4.88 \cdot 10^{-4}$$

$$\gamma_{mn} = 2\boldsymbol{\alpha}_n \cdot \boldsymbol{m} = 2 \cdot [3.2 \quad -5.2 \quad 2.2] \begin{bmatrix} 0 \\ -0.8 \\ -0.6 \end{bmatrix} \cdot 10^{-4} =$$

$$[6.4\cdot 0 + (-10.4)\cdot(-0.8) + 4.4\cdot(-0.6)] \cdot 10^{-4} = 5.68 \cdot 10^{-4}$$

c,

The $\boldsymbol{\alpha}_m$ strain vector can be calculated.

$$\boldsymbol{\alpha}_m = \boldsymbol{A}_P \cdot \boldsymbol{m} = \begin{bmatrix} 3 & 0 & -4 \\ 0 & -2 & 5 \\ -4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -0.8 \\ -0.6 \end{bmatrix} \cdot 10^{-4} =$$

$$= \begin{bmatrix} 3\cdot 0 + 0\cdot(-0.8) + (-4)\cdot(-0.6) \\ 0\cdot 0 + (-2)\cdot(-0.8) + 5\cdot(-0.6) \\ (-4)\cdot 0 + 5\cdot(-0.8) + 1\cdot(-0.6) \end{bmatrix} \cdot 10^{-4} = \begin{bmatrix} 2.4 \\ -1.4 \\ -4.6 \end{bmatrix} \cdot 10^{-4}$$

With the usage of $\boldsymbol{\alpha}_m$ strain vector the ε_m normal strain and the γ_{nm} shear angle can be calculated.

$$\varepsilon_m = \boldsymbol{\alpha}_m \cdot \boldsymbol{m} = [2.4 \quad -1.4 \quad -4.6] \begin{bmatrix} 0 \\ -0.8 \\ -0.6 \end{bmatrix} \cdot 10^{-4} =$$

$$= [2.4\cdot 0 + (-1.4)\cdot(-0.8) + (-4.6)\cdot(-0.6)] \cdot 10^{-4} = 3.88 \cdot 10^{-4}$$

$$\gamma_{nm} = 2 \cdot \boldsymbol{\alpha}_m \cdot \boldsymbol{n} = 2 \cdot [2.4 \quad -1.4 \quad -4.6] \begin{bmatrix} 0 \\ 0.6 \\ -0.8 \end{bmatrix} \cdot 10^{-4} =$$

$$= [4.8\cdot 0 + (-2.8)\cdot 0.6 + (-9.2)\cdot(-0.8)] \cdot 10^{-4} = 5.68 \cdot 10^{-4}$$

d,

$$\gamma_{mx} = 2 \cdot \boldsymbol{\alpha}_m \cdot \boldsymbol{i} = 2 \cdot [2.4 \quad -1.4 \quad -4.6] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \cdot 10^{-4} = 4.8 \cdot 10^{-4}$$

or

$$\gamma_{xm} = 2 \cdot \boldsymbol{\alpha}_x \cdot \boldsymbol{m} = 2 \cdot [3 \quad 0 \quad -4] \begin{bmatrix} 0 \\ -0.8 \\ -0.6 \end{bmatrix} \cdot 10^{-4} = 4.8 \cdot 10^{-4}$$

$$\gamma_{xy} = 2 \cdot \boldsymbol{\alpha}_x \cdot \boldsymbol{j} = 2 \cdot [3 \quad 0 \quad -4] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \cdot 10^{-4} = 0$$

Example 5

At point P the strain vectors are given in the reference xyz coordinate system. The unit vectors $\boldsymbol{n}$ and $\boldsymbol{m}$ are also known.

Data:

$$\boldsymbol{\alpha}_x = \varepsilon_x \boldsymbol{i} + \frac{1}{2}\gamma_{yx}\boldsymbol{j} + \frac{1}{2}\gamma_{zx}\boldsymbol{k} = (2\boldsymbol{i} - 4\boldsymbol{j}) \cdot 10^{-4}$$

$$\boldsymbol{\alpha}_y = \frac{1}{2}\gamma_{xy}\boldsymbol{i} + \varepsilon_y \boldsymbol{j} + \frac{1}{2}\gamma_{zy}\boldsymbol{k} = (-4\boldsymbol{i} + 5\boldsymbol{j} + 3\boldsymbol{k}) \cdot 10^{-4}$$

$$\boldsymbol{\alpha}_z = \frac{1}{2}\gamma_{xz}\boldsymbol{i} + \frac{1}{2}\gamma_{yz}\boldsymbol{j} + \varepsilon_z \boldsymbol{k} = (3\boldsymbol{j}) \cdot 10^{-4}$$

$$\boldsymbol{n} = -0.8\boldsymbol{i} + 0.6\boldsymbol{j}, \boldsymbol{m} = 0.6\boldsymbol{i} + 0.8\boldsymbol{j}$$

Questions:

a, Determine the strain tensor at body point P!

b, Determine the $\boldsymbol{\alpha}_n$ strain vector, the ε_n normal strain and the γ_{mn} shear angle!

Solution:

The strain tensor in general can be written with the following form:

$$\boldsymbol{A} = \begin{bmatrix} \varepsilon_x & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & \varepsilon_y & \frac{1}{2}\gamma_{yz} \\ \frac{1}{2}\gamma_{zx} & \frac{1}{2}\gamma_{zy} & \varepsilon_z \end{bmatrix} = [\boldsymbol{\alpha}_x \quad \boldsymbol{\alpha}_y \quad \boldsymbol{\alpha}_z]$$

Substituting the strain components from the strain vectors the strain tensor of body point P can be determined.

$$\boldsymbol{A}_P = \begin{bmatrix} 2 & -4 & 0 \\ -4 & 5 & 3 \\ 0 & 3 & 0 \end{bmatrix} \cdot 10^{-4}$$

The strain vector can be determined.

$$\boldsymbol{\alpha}_n = \boldsymbol{A} \cdot \boldsymbol{n} = \begin{bmatrix} 2 & -4 & 0 \\ -4 & 5 & 3 \\ 0 & 3 & 0 \end{bmatrix} \begin{bmatrix} -0.8 \\ 0.6 \\ 0 \end{bmatrix} \cdot 10^{-4} = \begin{bmatrix} -4 \\ 6.2 \\ 1.8 \end{bmatrix} \cdot 10^{-4}$$

$$= (-4\boldsymbol{i} + 6.2\boldsymbol{j} + 1.8\boldsymbol{k}) \cdot 10^{-4}$$

With the usage of $\boldsymbol{\alpha}_n$ strain vector the ε_n normal strain and the γ_{mn} shear angle can be calculated.

$$\varepsilon_n = \boldsymbol{\alpha}_n \cdot \boldsymbol{n} = \begin{bmatrix} -4 & 6.2 & 1.8 \end{bmatrix} \begin{bmatrix} -0.8 \\ 0.6 \\ 0 \end{bmatrix} \cdot 10^{-4} = (3.2 + 3.72) \cdot 10^{-4} = 6.92 \cdot 10^{-4}$$

$$\gamma_{mn} = 2\boldsymbol{\alpha}_n \cdot \boldsymbol{m} = 2 \cdot \begin{bmatrix} -4 & 6.2 & 1.8 \end{bmatrix} \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix} \cdot 10^{-4} =$$

$$= 2 \cdot (-2.4 + 4.96) \cdot 10^{-4} = 5.16 \cdot 10^{-4}$$

Example 6

At body point P in the coordinate system xyz the strain measures of an elastic solid body is known. The unit vector $\boldsymbol{n}$ is also known.

Data:

$$\varepsilon_x = 5 \cdot 10^{-3}, \quad \gamma_{yx} = \gamma_{xy} = 0$$

$$\varepsilon_y = 4 \cdot 10^{-3}, \quad \gamma_{yz} = \gamma_{zy} = 0$$

$$\varepsilon_z = 10^{-2}, \quad \gamma_{zx} = \gamma_{xz} = -10^{-2}$$

$$\boldsymbol{n} = 0.8\boldsymbol{i} + 0.6\boldsymbol{k}$$

Question:

Determine the strain state represented with the strain tensor of the body point P! Calculate the ε_n normal strain and the γ_{yn} shear angle!

Solution:

The strain tensor in general can be written with the following form, then substituting the given strain measures we get

$$\boldsymbol{A} = \begin{bmatrix} \varepsilon_x & \frac{1^{\gamma}}{2}\,xy & \frac{1^{\gamma}}{2}\,xz \\ \frac{1^{\gamma}}{2}\,yx & \varepsilon_y & \frac{1^{\gamma}}{2}\,yz \\ \frac{1^{\gamma}}{2}\,zx & \frac{1^{\gamma}}{2}\,zy & \varepsilon_z \end{bmatrix} = \begin{bmatrix} 5 & 0 & -5 \\ 0 & 4 & 0 \\ -5 & 0 & 10 \end{bmatrix} \cdot 10^{-3}$$

The normal strain can be calculated using the strain tensor at body point P.

$$\varepsilon_n = \boldsymbol{n} \cdot \boldsymbol{A} \cdot \boldsymbol{n} = [0.8 \quad 0 \quad 0.6] \begin{bmatrix} 5 & 0 & -5 \\ 0 & 4 & 0 \\ -5 & 0 & 10 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0 \\ 0.6 \end{bmatrix} \cdot 10^{-3} =$$

$$= [(3.2 - 2.4) + (-2.4 + 3.6)] \cdot 10^{-3} = 2 \cdot 10^{-3}$$

The shear angle can be calculated using the strain tensor at body point P.

$$\gamma_{yn} = 2\boldsymbol{j} \cdot \boldsymbol{A} \cdot \boldsymbol{n} = [0 \quad 2 \quad 0] \begin{bmatrix} 5 & 0 & -5 \\ 0 & 4 & 0 \\ -5 & 0 & 10 \end{bmatrix} \begin{bmatrix} 0.8 \\ 0 \\ 0.6 \end{bmatrix} \cdot 10^{-3} = 0$$

3. STRESS STATE OF SOLID BODIES

3.1. Theoretical background of the stress state of solid bodies

Stress tensor (state of stress)

Consider a general deformable body loaded by an equilibrium force system, see in Figure 1.1. Let's pass through this body by a hypothetical plane and neglect one part of the body. Both the remaining part and the neglected part of the body have to be in equilibrium. There is acting force system distributed along the plane which effects this equilibrium. Furthermore, this distributed force system must be equivalent to the force system acting on the neglected part of the body. Note that the interface is common and equal where the so called split is applied, see in Figure 3.1. The intensity vector of this distributed force system is called stress vector $\boldsymbol{\varrho}$. The unit vector of the cross section is $\boldsymbol{n}$. We can state that we know the state of stresses with respect to the point of interest, if we know the stress vector at any cross section. Note, if the body is cut by another arbitrary hypothetical plane going through the body, it will result in another distributed force system having the same properties discussed above.

The Newton's third law of action and reaction is satisfied and can be expressed by

$$\boldsymbol{\varrho}(-\boldsymbol{n}) = -\boldsymbol{\varrho}(\boldsymbol{n}). \tag{3.1}$$

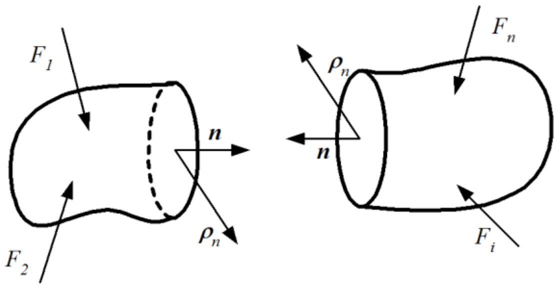


Figure 3.1. Introduction of the stress vector

The normal component is called normal stress σ_n, the component which is lying on the cross section is called shear stress τ. We can demonstrate the stress vectors belonging to the Cartesian coordinate system, thus we get the components of $\boldsymbol{\varrho}_x$, $\boldsymbol{\varrho}_y$ and $\boldsymbol{\varrho}_z$,

$$\begin{aligned}\boldsymbol{\varrho}_x &= \sigma_x \boldsymbol{i} + \tau_{yx} \boldsymbol{j} + \tau_{zx} \boldsymbol{k},\\ \boldsymbol{\varrho}_y &= \tau_{xy} \boldsymbol{i} + \sigma_y \boldsymbol{j} + \tau_{zy} \boldsymbol{k},\\ \boldsymbol{\varrho}_z &= \tau_{xz} \boldsymbol{i} + \tau_{yz} \boldsymbol{j} + \sigma_z \boldsymbol{k}.\end{aligned} \tag{3.2}$$

The first index in the case of shear stresses denotes the normal axis to the cross section, while the second one denotes the direction of the component. The normal components have only one subscript.

These components can be collected into a special tensor called stress tensor $\boldsymbol{T}$,

$$\boldsymbol{T} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = [\boldsymbol{\varrho}_x \quad \boldsymbol{\varrho}_y \quad \boldsymbol{\varrho}_z] = \boldsymbol{T}^T. \tag{3.3}$$

The stress tensor is symmetrical, that means the shear stresses at a point with reversed subscripts must be equal to each other, $\tau_{xy} = \tau_{yx}$, $\tau_{yz} = \tau_{zy}$ and $\tau_{zx} = \tau_{xz}$.

The stress tensor can be written in dyadic form using the stress vector

$$\boldsymbol{T} = \boldsymbol{\varrho}_x \circ \boldsymbol{i} + \boldsymbol{\varrho}_y \circ \boldsymbol{j} + \boldsymbol{\varrho}_z \circ \boldsymbol{k}. \tag{3.4}$$

Considering the Cauchy's theorem, the Cauchy stress vector can be determined

$$\boldsymbol{\varrho}_n = \boldsymbol{T}\boldsymbol{n}. \tag{3.5}$$

The generally used unit of the stress is $\frac{N}{mm^2} = MPa$.

3.2. Examples for the investigations of the stress state of solid bodies

Example 1

Determination of stress measures of solid body at a given body point!

The stress tensor of the body point P of a solid body is given!.

Data:

$$\boldsymbol{T}_P = \begin{bmatrix} 50 & -30 & 0 \\ -30 & 0 & 20 \\ 0 & 20 & 70 \end{bmatrix} MPa$$

Questions:

a, Determine the stress vectors in *x, y* and *z* directions!

b, Show the stress state on the so called elementary cube!

Solution:

a, The diadic form of the stress tensor is the following:

$$\boldsymbol{T}_P = \boldsymbol{\rho}_x \circ \boldsymbol{i} + \boldsymbol{\rho}_y \circ \boldsymbol{j} + \boldsymbol{\rho}_z \circ \boldsymbol{k}$$

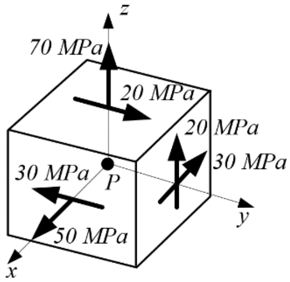
so the stress vector in the *x,y* and *z* directions, respectively:

$$\boldsymbol{\rho}_x = \sigma_x \boldsymbol{i} + \tau_{yx} \boldsymbol{j} + \tau_{zx} \boldsymbol{k} = (50\boldsymbol{i} - 30\boldsymbol{j}) MPa$$

$$\boldsymbol{\rho}_y = \tau_{xy} \boldsymbol{i} + \sigma_y \boldsymbol{j} + \tau_{zy} \boldsymbol{k} = (-30\boldsymbol{i} + 20\boldsymbol{k}) MPa$$

$$\boldsymbol{\rho}_z = \tau_{xz} \boldsymbol{i} + \tau_{yz} \boldsymbol{j} + \tau_{zx} \boldsymbol{k} = (20\boldsymbol{j} + 70\boldsymbol{k}) MPa$$

b, The stress state of the body point *P* on the elementary cube:



Example 2

Calculation of stress measures!

The $\boldsymbol{\rho}_x$, $\boldsymbol{\rho}_y$ and $\boldsymbol{\rho}_z$ stress vectors which belong to the directions $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ of a body point *P* and the $\boldsymbol{n}$ and $\boldsymbol{m}$ unit vectors are given.

Data:

$$\boldsymbol{\rho}_x = (60\boldsymbol{i} + 90\boldsymbol{k}) MPa, \boldsymbol{\rho}_y = \left(-30\sqrt{5}\boldsymbol{k}\right) MPa, \boldsymbol{\rho}_z = (90\boldsymbol{i} - 30\sqrt{5}\boldsymbol{j} + 50\boldsymbol{k}) MPa$$

$$\boldsymbol{n} = \frac{\sqrt{5}}{3}\boldsymbol{i} + \frac{2}{3}\boldsymbol{j}, \boldsymbol{m} = -\frac{2}{3}\boldsymbol{i} + \frac{\sqrt{5}}{3}\boldsymbol{j}$$

Questions:

a, Establish the $\boldsymbol{T}_P$ strain tensor of body point *P*!

b, Determine the σ_n normal strain in direction $\boldsymbol{n}$! Determine the τ_{mn} shear angle!

c, Determine the σ_m normal strain in direction $\boldsymbol{m}$! Determine the τ_{nm} shear angle!

d, Determine the τ_{zn} shear angle!

Solution:

a, The stress tensor in general and after substitution:

$$\boldsymbol{T}_P = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} 60 & 0 & 90 \\ 0 & 0 & -30\sqrt{5} \\ 90 & -30\sqrt{5} & 50 \end{bmatrix} MPa$$

b,

$$\sigma_n = \boldsymbol{n} \cdot \boldsymbol{T}_P \cdot \boldsymbol{n} = \boldsymbol{\rho}_n \cdot \boldsymbol{n} = [20\sqrt{5} \quad 0 \quad 10\sqrt{5}] \begin{bmatrix} \sqrt{5}/3 \\ 2/3 \\ 0 \end{bmatrix} MPa = \frac{100}{3} MPa$$

$$\boldsymbol{\rho}_n = \boldsymbol{T}_P \cdot \boldsymbol{n} = \begin{bmatrix} 60 & 0 & 90 \\ 0 & 0 & -30\sqrt{5} \\ 90 & -30\sqrt{5} & 50 \end{bmatrix} \begin{bmatrix} \sqrt{5}/3 \\ 2/3 \\ 0 \end{bmatrix} MPa = \begin{bmatrix} 20\sqrt{5} \\ 0 \\ 10\sqrt{5} \end{bmatrix} MPa$$

$$\tau_{mn} = \boldsymbol{m} \cdot \boldsymbol{T}_P \cdot \boldsymbol{n} = \boldsymbol{\rho}_n \cdot \boldsymbol{m} = [20\sqrt{5} \quad 0 \quad 10\sqrt{5}] \begin{bmatrix} -2/3 \\ \sqrt{5}/3 \\ 0 \end{bmatrix} MPa = -\frac{40\sqrt{5}}{3} MPa$$

c,

$$\sigma_m = \boldsymbol{m} \cdot \boldsymbol{T}_P \cdot \boldsymbol{m} = \boldsymbol{\rho}_m \cdot \boldsymbol{m} = [-40 \quad 0 \quad -110] \begin{bmatrix} -2/3 \\ \sqrt{5}/3 \\ 0 \end{bmatrix} MPa = \frac{80}{3} MPa$$

$$\boldsymbol{\rho}_m = \boldsymbol{T}_P \cdot \boldsymbol{m} = \begin{bmatrix} 60 & 0 & 90 \\ 0 & 0 & -30\sqrt{5} \\ 90 & -30\sqrt{5} & 50 \end{bmatrix} \begin{bmatrix} -2/3 \\ \sqrt{5}/3 \\ 0 \end{bmatrix} MPa = \begin{bmatrix} -40 \\ 0 \\ -110 \end{bmatrix} MPa$$

$$\tau_{nm} = \boldsymbol{n} \cdot \boldsymbol{T}_P \cdot \boldsymbol{m} = \boldsymbol{\rho}_m \cdot \boldsymbol{n} = [-40 \quad 0 \quad -110] \begin{bmatrix} \sqrt{5}/3 \\ 2/3 \\ 0 \end{bmatrix} MPa = -\frac{40\sqrt{5}}{3} MPa = \tau_{mn}$$

d,

$$\tau_{zn} = \boldsymbol{k} \cdot \boldsymbol{T}_p \cdot \boldsymbol{n} = \boldsymbol{\rho}_n \cdot \boldsymbol{k} = [20\sqrt{5} \quad 0 \quad 10\sqrt{5}] \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} MPa = 10\sqrt{5} MPa$$

Example 3

At point P the stress vectors are given in the reference xyz coordinate system. The unit vectors $\boldsymbol{n}$ and $\boldsymbol{m}$ are also known.

Data:

$$\boldsymbol{\varrho}_x = \sigma_x \boldsymbol{i} + \tau_{yx} \boldsymbol{j} + \tau_{zx} \boldsymbol{k} = (20\boldsymbol{i} - 40\boldsymbol{j}) MPa$$

$$\boldsymbol{\varrho}_y = \tau_{xy} \boldsymbol{i} + \sigma_y \boldsymbol{j} + \tau_{zy} \boldsymbol{k} = (-40\boldsymbol{i} + 50\boldsymbol{j} + 30\boldsymbol{k}) MPa$$

$$\varrho_z = \tau_{xz}\boldsymbol{i} + \tau_{yz}\boldsymbol{j} + \sigma_z\boldsymbol{k} = (30\boldsymbol{j})MPa$$

$$\boldsymbol{n} = -0.8\boldsymbol{i} + 0.6\boldsymbol{j}, \boldsymbol{m} = 0.6\boldsymbol{i} + 0.8\boldsymbol{j}$$

Questions:

a, Determine the stress tensor at body point *P*!

b, Determine the $\boldsymbol{\rho}_n$ stress vector, the σ_n normal stress and the τ_{mn} shear stress!

Solution:

The stress tensor in general can be written with the following form:

$$\boldsymbol{T} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = [\varrho_x \quad \varrho_y \quad \varrho_z]$$

Substituting the stress components from the stress vectors the stress tensor of body point *P* can be determined.

$$\boldsymbol{T}_P = \begin{bmatrix} 20 & -40 & 0 \\ -40 & 50 & 30 \\ 0 & 30 & 0 \end{bmatrix} MPa$$

The Cauchy stress vector can be determined.

$$\varrho_n = \boldsymbol{T} \cdot \boldsymbol{n} = \begin{bmatrix} 20 & -40 & 0 \\ -40 & 50 & 30 \\ 0 & 30 & 0 \end{bmatrix} \begin{bmatrix} -0.8 \\ 0.6 \\ 0 \end{bmatrix} = \begin{bmatrix} -40 \\ 62 \\ 18 \end{bmatrix} MPa = (-40\boldsymbol{i} + 62\boldsymbol{j} + 18\boldsymbol{k})MPa$$

With the usage of ϱ_n stress vector the σ_n normal stress and the τ_{mn} shear stress can be calculated.

$$\sigma_n = \varrho_n \cdot \boldsymbol{n} = [-40 \quad 62 \quad 18] \begin{bmatrix} -0.8 \\ 0.6 \\ 0 \end{bmatrix} = 32 + 37.2 = 69.2MPa$$

$$\tau_{mn} = \varrho_n \cdot \boldsymbol{m} = [-40 \quad 62 \quad 18] \begin{bmatrix} 0.6 \\ 0.8 \\ 0 \end{bmatrix} = -24 + 49.6 = 25.6MPa$$

Example 4

The ϱ_x, ϱ_y and ϱ_z stress vectors are given of a solid body point *P* in the *x*,*y* and *z* directions, respectively. The unit vectors $\boldsymbol{m}$ and $\boldsymbol{n}$ denote two directions and are perpendicular to each other.

Data:

$$\varrho_x = (3\boldsymbol{i} - 4\boldsymbol{k})MPa$$

$$\boldsymbol{\varrho}_y = (-2\boldsymbol{j} + 5\boldsymbol{k})MPa$$

$$\boldsymbol{\varrho}_z = (-4\boldsymbol{i} + 5\boldsymbol{j} + \boldsymbol{k})MPa$$

$$\boldsymbol{m} = -0.8\boldsymbol{j} - 0.6\boldsymbol{k}, \boldsymbol{n} = 0.6\boldsymbol{j} - 0.8\boldsymbol{k}$$

Questions:

a, Using the stress vectors establish the $\boldsymbol{T}_P$ strain tensor!

b, Determine the $\boldsymbol{\varrho}_n$ stress vector in $\boldsymbol{n}$ direction and the σ_n normal stress and the τ_{mn} shear stress!

c, Determine the $\boldsymbol{\alpha}_m$ stress vector in $\boldsymbol{m}$ direction and the σ_m normal stress and the τ_{nm} shear stress!

d, Determine the τ_{mx} and τ_{xy} shear stresses on the given planes!

Solution:

a, The stress tensor in general and after substitution:

$$\boldsymbol{T}_P = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_{zx} & \tau_{zy} & \sigma_z \end{bmatrix} = \begin{bmatrix} 3 & 0 & -4 \\ 0 & -2 & 5 \\ -4 & 5 & 1 \end{bmatrix} MPa$$

b,

$$\sigma_n = \boldsymbol{n} \cdot \boldsymbol{T}_P \cdot \boldsymbol{n} = \boldsymbol{\varrho}_n \cdot \boldsymbol{n}$$

The $\boldsymbol{\varrho}_n$ stress vector can be calculated.

$$\boldsymbol{\varrho}_n = \boldsymbol{T}_P \cdot \boldsymbol{n} = \begin{bmatrix} 3 & 0 & -4 \\ 0 & -2 & 5 \\ -4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0.6 \\ -0.8 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 \cdot 0 + 0 \cdot 0.6 + (-4) \cdot (-0.8) \\ 0 \cdot 0 + (-2) \cdot 0.6 + 5 \cdot (-0.8) \\ (-4) \cdot 0 + 5 \cdot 0.6 + 1 \cdot (-0.8) \end{bmatrix} = \begin{bmatrix} 3.2 \\ -5.2 \\ 2.2 \end{bmatrix} MPa$$

With the usage of $\boldsymbol{\varrho}_n$ strain vector the σ_n normal stress and the τ_{mn} shear stress can be calculated.

$$\sigma_n = \boldsymbol{\varrho}_n \cdot \boldsymbol{n} = [3.2 \quad -5.2 \quad 2.2] \begin{bmatrix} 0 \\ 0.6 \\ -0.8 \end{bmatrix} =$$

$$= [3.2 \cdot 0 + (-5.2) \cdot 0.6 + 2.2 \cdot (-0.8)] = -4.88MPa$$

$$\tau_{mn} = \boldsymbol{\varrho}_n \cdot \boldsymbol{m} = [3.2 \quad -5.2 \quad 2.2] \begin{bmatrix} 0 \\ -0.8 \\ -0.6 \end{bmatrix} =$$

$$[3.2 \cdot 0 + (-5.2) \cdot (-0.8) + 2.2 \cdot (-0.6)] = 2.84 MPa$$

c,

The $\boldsymbol{\varrho}_m$ stress vector can be calculated.

$$\boldsymbol{\varrho}_m = \boldsymbol{T}_P \cdot \boldsymbol{m} = \begin{bmatrix} 3 & 0 & -4 \\ 0 & -2 & 5 \\ -4 & 5 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ -0.8 \\ -0.6 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 \cdot 0 + 0 \cdot (-0.8) + (-4) \cdot (-0.6) \\ 0 \cdot 0 + (-2) \cdot (-0.8) + 5 \cdot (-0.6) \\ (-4) \cdot 0 + 5 \cdot (-0.8) + 1 \cdot (-0.6) \end{bmatrix} = \begin{bmatrix} 2.4 \\ -1.4 \\ -4.6 \end{bmatrix} MPa$$

With the usage of $\boldsymbol{\varrho}_m$ stress vector the σ_m normal stress and the τ_{nm} shear stress can be calculated.

$$\sigma_m = \boldsymbol{\varrho}_m \cdot \boldsymbol{m} = [2.4 \quad -1.4 \quad -4.6] \begin{bmatrix} 0 \\ -0.8 \\ -0.6 \end{bmatrix} =$$

$$= [2.4 \cdot 0 + (-1.4) \cdot (-0.8) + (-4.6) \cdot (-0.6)] = 3.88 MPa$$

$$\tau_{nm} = \boldsymbol{\varrho}_m \cdot \boldsymbol{n} = [2.4 \quad -1.4 \quad -4.6] \begin{bmatrix} 0 \\ 0.6 \\ -0.8 \end{bmatrix} =$$

$$= [2.4 \cdot 0 + (-1.4) \cdot 0.6 + (-4.6) \cdot (-0.8)] = 2.84 MPa$$

d,

$$\tau_{mx} = \boldsymbol{\varrho}_m \cdot \boldsymbol{i} = [2.4 \quad -1.4 \quad -4.6] \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = 2.4 MPa$$

or

$$\tau_{xm} = \boldsymbol{\varrho}_x \cdot \boldsymbol{m} = [3 \quad 0 \quad -4] \begin{bmatrix} 0 \\ -0.8 \\ -0.6 \end{bmatrix} = 2.4 MPa$$

$$\tau_{xy} = \boldsymbol{\varrho}_x \cdot \boldsymbol{j} = [3 \quad 0 \quad -4] \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 0$$

Example 5

The stress vector of a body point P is known.

Data:

$$\boldsymbol{\varrho}_n = (581\boldsymbol{i} - 100\boldsymbol{j} + 200\boldsymbol{k})\frac{N}{mm^2} = (581\boldsymbol{i} - 100\boldsymbol{j} + 200\boldsymbol{k})MPa$$

$$\boldsymbol{n} = 0.5\boldsymbol{i} + 0.5\boldsymbol{j} + \frac{\sqrt{2}}{2}\boldsymbol{k}, \qquad \boldsymbol{m} = -0.5\boldsymbol{i} - 0.5\boldsymbol{j} + \frac{\sqrt{2}}{2}\boldsymbol{k}, \quad \boldsymbol{l} = \frac{\sqrt{2}}{2}\boldsymbol{i} - \frac{\sqrt{2}}{2}\boldsymbol{j}$$

Question:
Determine the components of the stress vector!
Solution:
The stress vector can be written in general form as follows:

$$\boldsymbol{\varrho}_n = \sigma_n \cdot \boldsymbol{n} + \boldsymbol{\tau}_n$$

where $\boldsymbol{\varrho}_n$ is the stress vector, σ_n is the normal stress and $\boldsymbol{\tau}_n$ is the shear stress vector.
The unit vectors can be written in matrix form as follows:

$$\boldsymbol{n} = \begin{bmatrix} 0.5 \\ 0.5 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \qquad \boldsymbol{m} = \begin{bmatrix} -0.5 \\ -0.5 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \qquad \boldsymbol{l} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix}$$

It is also important to control if the given vectors are unit vectors or not.

$$|\boldsymbol{n}| = \sqrt{0.5^2 + 0.5^2 + \left(\frac{\sqrt{2}}{2}\right)^2} = 1; |\boldsymbol{m}| = 1; \ |\boldsymbol{l}| = 1$$

It can be stated that the given $\boldsymbol{n}$, $\boldsymbol{m}$ and $\boldsymbol{l}$ vectors are unit vectors.

The shear stress vector can be written with its components.

$$\boldsymbol{\tau}_n = \tau_{mn} \cdot \boldsymbol{m} + \tau_{lm} \cdot \boldsymbol{l}$$

The normal stress vector has only one component.

$$\boldsymbol{\sigma_n} = \sigma_n \cdot \boldsymbol{n}$$

The normal stress component can be calculated.

$$\sigma_n = \boldsymbol{\sigma_n} \cdot \boldsymbol{n} = [581 \quad -100 \quad 200] \begin{bmatrix} 0.5 \\ 0.5 \\ \frac{\sqrt{2}}{2} \end{bmatrix} = 290.5 - 50 + 141.42 = 381.92 \ MPa$$

The shear stress components can be calculated.

$$\tau_{mn} = \boldsymbol{\varrho_n} \cdot \boldsymbol{m} = [581 \quad -100 \quad 200] \begin{bmatrix} -0.5 \\ -0.5 \\ \frac{\sqrt{2}}{2} \end{bmatrix} = -290.5 + 50 + 141.2 = -99.08MP$$

$$\tau_{ln} = \boldsymbol{\varrho_n} \cdot \boldsymbol{l} = [581 \quad -100 \quad 200] \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = 410.82 + 70.71 = 481.53\, MPa$$

Example 6

The stress tensor of a solid body point P is known. The unit vector $\boldsymbol{n}$ is also known.

Data:

$$\boldsymbol{T}_p = \begin{bmatrix} 400 & -500 & 0 \\ -500 & 0 & 500 \\ 0 & 500 & -500 \end{bmatrix} MPa$$

$$\boldsymbol{n} = 0.8\boldsymbol{j} + 0.6\boldsymbol{k}$$

Questions:

a, Determine the $\boldsymbol{\varrho}_x$, $\boldsymbol{\varrho}_y$ and $\boldsymbol{\varrho}_z$ stress vectors related to $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ directions, respectively!

b, Determine the $\boldsymbol{\varrho}_n$ stress vector, the σ_n normal stress and the τ_n shear stress!

Solution:

The stress tensor in general form can be written with its stress components.

$$\boldsymbol{T} = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & \sigma_y & \tau_{yz} \\ \tau_x & \tau_{zy} & \sigma_z \end{bmatrix} = [\boldsymbol{\varrho}_x \quad \boldsymbol{\varrho}_y \quad \boldsymbol{\varrho}_z]$$

The $\boldsymbol{\varrho}_x$, $\boldsymbol{\varrho}_y$ and $\boldsymbol{\varrho}_z$ stress vectors related to $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ directions can be written using the stress tensor.

$$\boldsymbol{\varrho_x} = \sigma_x\boldsymbol{i} + \tau_{yx}\boldsymbol{j} + \tau_{zx}\boldsymbol{k} = (400\boldsymbol{i} + 500\boldsymbol{j})MPa$$

$$\boldsymbol{\varrho_y} = \tau_{xy}\boldsymbol{i} + \sigma_y\boldsymbol{j} + \tau_{zy}\boldsymbol{k} = (-500\boldsymbol{i} + 500\boldsymbol{k})MPa$$

$$\boldsymbol{\varrho_z} = \tau_{xz}\boldsymbol{i} + \tau_{yz}\boldsymbol{j} + \sigma_z\boldsymbol{k} = (500\boldsymbol{j} - 500\boldsymbol{k})MPa$$

b, The stress vector can be calculated using the stress tensor as follows:

$$\boldsymbol{\varrho}_n = \boldsymbol{T} \cdot \boldsymbol{n} = \begin{bmatrix} 400 & -500 & 0 \\ -500 & 0 & 500 \\ 0 & 500 & -500 \end{bmatrix} \begin{bmatrix} 0 \\ 0.8 \\ 0.6 \end{bmatrix} = \begin{bmatrix} -400 \\ 300 \\ 100 \end{bmatrix} MPa$$

The stress vector in vector form

$$\boldsymbol{\varrho}_n = (-400\boldsymbol{i} + 300\boldsymbol{j} + 100\boldsymbol{k})MPa$$

The components of the stress vector can be determined as follows:

$$\sigma_n = \boldsymbol{n} \cdot \boldsymbol{T} \cdot \boldsymbol{n} = \boldsymbol{\varrho}_n \cdot \boldsymbol{n} = [-400 \quad 300 \quad 100] \begin{bmatrix} 0 \\ 1.8 \\ 0.6 \end{bmatrix} = 300\ MPa$$

$$\boldsymbol{\varrho}_n = \sigma_n \cdot \boldsymbol{n} + \boldsymbol{\tau}_n \rightarrow \boldsymbol{\tau}_n = \boldsymbol{\varrho}_n - \sigma_n \cdot \boldsymbol{n} = \begin{bmatrix} -400 \\ 300 \\ 100 \end{bmatrix} - \begin{bmatrix} 0 \\ 240 \\ 180 \end{bmatrix} = \begin{bmatrix} -400 \\ 60 \\ -80 \end{bmatrix} MPa$$

Example 7

The stress tensor in a given point is known. The $\boldsymbol{n}$ and $\boldsymbol{m}$ unit vectors are also known!

Data:

$$\boldsymbol{T} = \begin{bmatrix} 80 & 0 & 0 \\ 0 & 40 & -32 \\ 0 & -32 & -80 \end{bmatrix} MPa$$

$$\boldsymbol{n} = \frac{1}{\sqrt{17}}(4\boldsymbol{j} - \boldsymbol{k}), \quad \boldsymbol{m} = \frac{1}{\sqrt{17}}(\boldsymbol{j} - 4\boldsymbol{k})$$

Questions:

a, Establish the stress tensor in diadic form!

b, Calculate the σ_n and σ_m normal stresses on the planes given with its normal unit vectors $\boldsymbol{n}$ and $\boldsymbol{m}$!

Solution:

a, The stress tensor in diadic form is the following

$$\boldsymbol{T} = \boldsymbol{\varrho}_x \circ \boldsymbol{i} + \boldsymbol{\varrho}_y \circ \boldsymbol{j} + \boldsymbol{\varrho}_z \circ \boldsymbol{k}$$

After substitution we get

$$\boldsymbol{T} = 80\boldsymbol{i} \circ \boldsymbol{i} + (40\boldsymbol{j} - 32\boldsymbol{k}) \circ \boldsymbol{j} + (-32\boldsymbol{j} - 80\boldsymbol{k}) \circ \boldsymbol{k}$$

b, The normal stresses can be calculated as follows:

$$\sigma_n = \boldsymbol{\varrho}_n \cdot \boldsymbol{n} = \boldsymbol{n} \cdot T \cdot \boldsymbol{n} = \begin{bmatrix} 0 & \frac{4}{\sqrt{17}} & -\frac{1}{\sqrt{17}} \end{bmatrix} \begin{bmatrix} 80 & 0 & 0 \\ 0 & 40 & -32 \\ 0 & -32 & -80 \end{bmatrix} \begin{bmatrix} 0 \\ \frac{4}{\sqrt{17}} \\ -\frac{1}{\sqrt{17}} \end{bmatrix} =$$

$$==\left(\frac{640}{17}+\frac{128}{17}\right)+\left(\frac{128}{17}-\frac{80}{17}\right)=48\ MPa$$

$$\sigma_m=\boldsymbol{\varrho}_n\cdot\boldsymbol{m}=\boldsymbol{m}\cdot\boldsymbol{T}\cdot\boldsymbol{m}=\begin{bmatrix}0 & \frac{1}{\sqrt{17}} & \frac{4}{\sqrt{17}}\end{bmatrix}\begin{bmatrix}80 & 0 & 0\\ 0 & 40 & -32\\ 0 & -32 & -80\end{bmatrix}\begin{bmatrix}0\\ \frac{1}{\sqrt{17}}\\ \frac{4}{\sqrt{17}}\end{bmatrix}=$$

$$=\left(\frac{40}{17}-\frac{128}{17}\right)+\left(-\frac{128}{17}-\frac{1280}{17}\right)=-\frac{1496}{17}=-88MPa$$

Example 8

The stress state of an elastic body in body point P is known with stress tensor. The $\boldsymbol{l}$, $\boldsymbol{m}$ and $\boldsymbol{n}$ unit vectors are also known and they are perpendicular to each other.

Data:

$$\boldsymbol{T}_P=\begin{bmatrix}50 & 20 & -40\\ 20 & 80 & 30\\ -40 & 30 & -20\end{bmatrix}MPa$$

$$\boldsymbol{l}=\frac{2}{3}\boldsymbol{i}-\frac{2}{3}\boldsymbol{j}+\frac{1}{3}\boldsymbol{k},\quad \boldsymbol{m}=-\frac{2}{3}\boldsymbol{i}-\frac{1}{3}\boldsymbol{j}+\frac{2}{3}\boldsymbol{k},\quad \boldsymbol{n}=\frac{1}{3}\boldsymbol{i}+\frac{2}{3}\boldsymbol{j}+\frac{2}{3}\boldsymbol{k}$$

$$|\boldsymbol{l}|=|\boldsymbol{m}|=|\boldsymbol{n}|=1,\quad \boldsymbol{l}\cdot\boldsymbol{m}=\boldsymbol{m}\cdot\boldsymbol{n}=\boldsymbol{n}\cdot\boldsymbol{l}=0$$

Questions:

a, Determine the $\boldsymbol{\varrho}_x$, $\boldsymbol{\varrho}_y$ and $\boldsymbol{\varrho}_z$ stress vectors!

b, Determine the $\boldsymbol{\varrho}_n$ stress vector, the σ_n normal stress and the τ_{mn} and τ_{ln} shear stresses!

Solution:

a, The stress tensor in general form is

$$\boldsymbol{T}=\begin{bmatrix}\sigma_x & \tau_{xy} & \tau_{xz}\\ \tau_{yx} & \sigma_y & \tau_{yz}\\ \tau_{zx} & \tau_{zy} & \sigma_z\end{bmatrix}=[\boldsymbol{\varrho}_x\quad \boldsymbol{\varrho}_y\quad \boldsymbol{\varrho}_z]$$

so the stress vector in x,y and z directions can be calculated.

$$\boldsymbol{\varrho}_x=(50\boldsymbol{i}+20\boldsymbol{j}-40\boldsymbol{k})MPa$$

$$\boldsymbol{\varrho}_y=(20\boldsymbol{i}+80\boldsymbol{j}-30\boldsymbol{k})MPa$$

$$\boldsymbol{\varrho}_z=(-40\boldsymbol{i}+30\boldsymbol{j}-20\boldsymbol{k})MPa$$

b, The $\boldsymbol{\varrho}_n$ stress vector can be also calculated using the stress tensor and the related unit vector.

$$\boldsymbol{\varrho}_n = \boldsymbol{T}_P \cdot \boldsymbol{n} = \begin{bmatrix} 50 & 20 & -40 \\ 20 & 80 & 30 \\ -40 & 30 & -20 \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} \frac{10}{3} \\ 80 \\ -\frac{20}{3} \end{bmatrix} MPa$$

The components of the $\boldsymbol{\varrho}_n$ stress vector can be determined.

$$\sigma_n = \sigma_n = \boldsymbol{n} \cdot \boldsymbol{T}_P \cdot \boldsymbol{n} = \boldsymbol{\varrho}_n \cdot \boldsymbol{n} = \begin{bmatrix} \frac{10}{3} & 80 & -\frac{20}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{2}{3} \end{bmatrix} = 50\ MPa$$

$$\tau_{mn} = \boldsymbol{m} \cdot \boldsymbol{T}_P \cdot \boldsymbol{n} = \boldsymbol{\varrho}_n \cdot \boldsymbol{m} = \begin{bmatrix} \frac{10}{3} & 80 & -\frac{20}{3} \end{bmatrix} \cdot \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix} = -\frac{100}{3}\ MPa$$

$$\tau_{ln} = \boldsymbol{l} \cdot \boldsymbol{T}_P \cdot \boldsymbol{n} = \boldsymbol{\varrho}_n \cdot \boldsymbol{l} = \begin{bmatrix} \frac{10}{3} & 80 & -\frac{20}{3} \end{bmatrix} \cdot \begin{bmatrix} \frac{2}{3} \\ -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix} = -\frac{160}{3}\ MPa$$

Example 9

The $\boldsymbol{\varrho}_n$ stress vector and the $\boldsymbol{n}$ unit vectors are known of an elastic solid body.

Data:

$$\boldsymbol{\varrho}_n = (-400\boldsymbol{i} + 300\boldsymbol{j} + 200\boldsymbol{k})\ MPa, \quad \boldsymbol{n} = 0.8\boldsymbol{i} + 0.6\boldsymbol{k},$$

Question:

Determine the σ_n normal stress and the $\boldsymbol{\tau}_n$ shear stress vector!

Solution:

The normal stress is

$$\sigma_n = \boldsymbol{\varrho}_n \cdot \boldsymbol{n} = [-400 \quad 300 \quad 200] \cdot \begin{bmatrix} 0.8 \\ 0 \\ 0.6 \end{bmatrix} = -200\ MPa$$

The shear stress vector is

$$\boldsymbol{\tau}_n = \boldsymbol{\varrho}_n - \sigma_n \cdot \boldsymbol{n} = (-400\boldsymbol{i} + 300\boldsymbol{j} + 200\boldsymbol{k}) - (-200) \cdot (0.8\boldsymbol{i} + 0.6\boldsymbol{k}) =$$

$$= (-240\boldsymbol{i} + 300\boldsymbol{j} + 320\boldsymbol{k})\ MPa$$

4. CONSTITUTIVE EQUATION

4.1. Theoretical background of the constitutive equation of the linear elasticity

Constitutive equations

Now let's investigate the material response. Relations characterizing the physical properties of materials are called constitutive equations. Wide variety of materials is used, so the development of constitutive equations is one of the most researched fields in mechanics. The constitutive laws are developed through empirical relations based on experiments. The mechanical behaviour of solid materials is defined by stress-strain relations. The relations express the stress as a function of the strain. In this case the elastic solid material model is chosen. This model describes a deformable solid continuum that recovers its original shape when the loadings causing the deformation are removed. Furthermore we investigate only the constitutive law which is linear leading to a linear elastic solid. Many structural materials including metals, plastics, wood, concrete exhibit linear elastic behavior under small deformations. The stress-strain relation of a linear elastic material can be seen in Figure 4.1.

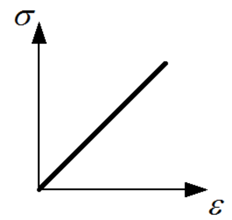


Figure 4.1. Linear elastic material model

The well known Hooke's law is valid for linear elastic material description. In the case of single axis stress state the simplified Hooke's law is

$$\sigma = E\varepsilon, \tag{4.1}$$

where E is the Young's modulus, which is a real material constants. In the case of torsion the simplified Hooke's law is

$$\tau = G\gamma, \tag{4.2}$$

where G is the shear modulus. Considering linear elastic material

$$E = 2G(1 + \nu) \tag{4.3}$$

is valid, where ν is the Poisson ratio which describes the relationship between elongation and contraction.

In general case the Hooke's law can be expressed as tensor equations which is valid for arbitrary coordinate systems,

$$\boldsymbol{T} = 2G\left(\boldsymbol{A} + \frac{\nu}{1-2\nu}A_I\boldsymbol{I}\right), \tag{4.4}$$

or

$$\boldsymbol{A} = \frac{1}{2G}\left(\boldsymbol{T} - \frac{\nu}{1+\nu}T_I\boldsymbol{I}\right), \tag{4.5}$$

where A_I is the first scalar invariant of the strain tensor,

$$A_I = \varepsilon_x + \varepsilon_y + \varepsilon_z = \varepsilon_1 + \varepsilon_2 + \varepsilon_3. \tag{4.6}$$

The Equation 4.4. can be written in scalar notation and we get

$$\begin{aligned}
\sigma_x &= 2G\left[\varepsilon_x + \frac{\nu}{1-2\nu}\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)\right], \\
\sigma_y &= 2G\left[\varepsilon_y + \frac{\nu}{1-2\nu}\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)\right], \\
\sigma_z &= 2G\left[\varepsilon_z + \frac{\nu}{1-2\nu}\left(\varepsilon_x + \varepsilon_y + \varepsilon_z\right)\right], \\
\tau_{xy} &= G\gamma_{xy}, \\
\tau_{yz} &= G\gamma_{yz}, \\
\tau_{zx} &= G\gamma_{zx}.
\end{aligned} \tag{4.7}$$

The Equation 4.5. can be written in scalar notation and we get

$$\begin{aligned}
\varepsilon_x &= \frac{1}{2G}\left[\sigma_x - \frac{\nu}{1+\nu}\left(\sigma_x + \sigma_y + \sigma_z\right)\right], \\
\varepsilon_y &= \frac{1}{2G}\left[\sigma_y - \frac{\nu}{1+\nu}\left(\sigma_x + \sigma_y + \sigma_z\right)\right], \\
\varepsilon_z &= \frac{1}{2G}\left[\sigma_z - \frac{\nu}{1+\nu}\left(\sigma_x + \sigma_y + \sigma_z\right)\right], \\
\gamma_{xy} &= \frac{\tau_{xy}}{G}, \\
\gamma_{yz} &= \frac{\tau_{yz}}{G}, \\
\gamma_{zx} &= \frac{\tau_{zx}}{G}.
\end{aligned} \tag{4.8}$$

For isotropic linear elastic material these relations can be described in matrix form,

$$\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{bmatrix}=\frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix}1-\nu & \nu & \nu & 0 & 0 & 0\\ \nu & 1-\nu & \nu & 0 & 0 & 0\\ \nu & \nu & 1-\nu & 0 & 0 & 0\\ 0 & 0 & 0 & 1-2\nu & 0 & 0\\ 0 & 0 & 0 & 0 & 1-2\nu & 0\\ 0 & 0 & 0 & 0 & 0 & 1-2\nu\end{bmatrix}\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{bmatrix}, \qquad (4.9)$$

or

$$\begin{bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{bmatrix}=\begin{bmatrix}\frac{1}{E} & -\frac{\nu}{E} & -\frac{\nu}{E} & 0 & 0 & 0\\ -\frac{\nu}{E} & \frac{1}{E} & -\frac{\nu}{E} & 0 & 0 & 0\\ -\frac{\nu}{E} & -\frac{\nu}{E} & \frac{1}{E} & 0 & 0 & 0\\ 0 & 0 & 0 & \frac{1}{G} & 0 & 0\\ 0 & 0 & 0 & 0 & \frac{1}{G} & 0\\ 0 & 0 & 0 & 0 & 0 & \frac{1}{G}\end{bmatrix}\begin{bmatrix}\sigma_x\\ \sigma_y\\ \sigma_z\\ \tau_{xy}\\ \tau_{yz}\\ \tau_{zx}\end{bmatrix}. \qquad (4.10)$$

4.2. Examples for the investigations of the Hooke's law

Example 1

The usage of the general Hooke's for the determination of the stress tensor!

The strain state of a body point P is given. With the usage of the general Hooke's law determine the stress tensor of body point P.

Data:

$$\boldsymbol{\alpha}_x=[4 \quad 3 \quad 0]\cdot 10^{-5}, \boldsymbol{\alpha}_y=[3 \quad -2 \quad -3]\cdot 10^{-5}, \boldsymbol{\alpha}_z=[0 \quad -3 \quad 7]\cdot 10^{-5}$$

$$\nu=0.3;\ \ E=2.1\cdot 10^5 MPa$$

Solution:

The form of the general Hooke's law:

$$\boldsymbol{T}_P=2G\left(\boldsymbol{A}_P+\frac{\nu}{1-2\nu}A_I\cdot\boldsymbol{I}\right),$$

where

$$E = 2G(1+\nu) \Rightarrow 2G = \frac{E}{1+\nu} = 1.6 \cdot 10^5 MPa$$

The strain tensor can be determined from the strain vectors.

$$\boldsymbol{A}_P = \begin{bmatrix} 4 & 3 & 0 \\ 3 & -2 & -3 \\ 0 & -3 & 7 \end{bmatrix} 10^{-5}$$

The first scalar invariant of the strain tensor can be calculated from the strain tensor.

$$A_I = \varepsilon_x + \varepsilon_y + \varepsilon_z = 9 \cdot 10^{-5}$$

The unit tensor is

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The stress tensor can be determined as follows:

$$\boldsymbol{T}_P = 1.6 \cdot 10^5 MPa \left(\begin{bmatrix} 4 & 3 & 0 \\ 3 & -2 & -3 \\ 0 & -3 & 7 \end{bmatrix} 10^{-5} + \frac{0.3}{1 - 2 \cdot 0.3} \cdot 9 \cdot 10^{-5} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right) =$$

$$= 1.6 \cdot 10^5 MPa \left(\begin{bmatrix} 4 & 3 & 0 \\ 3 & -2 & -3 \\ 0 & -3 & 7 \end{bmatrix} 10^{-5} + \begin{bmatrix} 6.75 & 0 & 0 \\ 0 & 6.75 & 0 \\ 0 & 0 & 6.75 \end{bmatrix} 10^{-5} \right) =$$

$$= 1.6 \cdot 10^5 MPa \begin{bmatrix} 10.75 & 3 & 0 \\ 3 & 4.75 & -3 \\ 0 & -3 & 13.75 \end{bmatrix} 10^{-5} = \begin{bmatrix} 17.2 & 4.8 & 0 \\ 4,8 & 7.6 & -4.8 \\ 0 & -4.8 & 22 \end{bmatrix} MPa$$

Example 2

The usage of the general Hooke's for the determination of the strain tensor!

The $\boldsymbol{\varrho}_x$, $\boldsymbol{\varrho}_y$ and $\boldsymbol{\varrho}_z$ stress vectors which belong to the directions $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ of a body point P are given. With the usage of the general Hooke's law determine the strain tensor of body point P.

Data:

$$\boldsymbol{\varrho}_x = [20 \quad 30 \quad -30] MPa$$

$$\boldsymbol{\varrho}_y = [30 \quad -40 \quad 0] MPa$$

$$\boldsymbol{\varrho}_z = [-30 \quad 0 \quad -80] MPa$$

$$\nu = 0.3; \quad E = 2.1 \cdot 10^5 MPa$$

Solution:

The form of the general Hooke's law:

$$\boldsymbol{A}_P = \frac{1}{2G}\left(\boldsymbol{T}_P - \frac{\nu}{1+\nu} T_I \cdot \boldsymbol{I}\right),$$

where

$$E = 2G(1+\nu) \rightarrow 2G = \frac{E}{1+\nu} = 1.6 \cdot 10^5 MPa$$

The stress tensor can be determined from the strain vectors.

$$\boldsymbol{T}_P = \begin{bmatrix} 20 & 30 & -30 \\ 30 & -40 & 0 \\ -30 & 0 & -80 \end{bmatrix} MPa$$

The first scalar invariant of the stress tensor can be calculated from the stress tensor.

$$T_I = \sigma_x + \sigma_y + \sigma_z = -100 MPa$$

The unit tensor is

$$\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The strain tensor can be determined as follows:

$$\boldsymbol{A}_P = \frac{1}{1.6 \cdot 10^5}\left(\begin{bmatrix} 20 & 30 & -30 \\ 30 & -40 & 0 \\ -30 & 0 & -80 \end{bmatrix} - \frac{0,3}{1+0,3}(-100)\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) =$$

$$= \frac{1}{1.6 \cdot 10^5}\left(\begin{bmatrix} 20 & 30 & -30 \\ 30 & -40 & 0 \\ -30 & 0 & -80 \end{bmatrix} - \begin{bmatrix} -23.08 & 0 & 0 \\ 0 & -23.08 & 0 \\ 0 & 0 & -23.08 \end{bmatrix}\right) =$$

$$= \frac{1}{1.6 \cdot 10^5 MPa}\begin{bmatrix} 43.08 & 30 & -30 \\ 30 & -63.08 & 0 \\ -30 & 0 & -103.08 \end{bmatrix} MPa =$$

$$\boldsymbol{A}_P = \begin{bmatrix} 26.88 & 18.75 & -18.75 \\ 18.75 & -39.43 & 0 \\ -18.75 & 0 & -64.43 \end{bmatrix} \cdot 10^{-5}$$

5. SIMPLE LOADINGS - TENSION AND COMPRESSION

5.1. Theoretical background of simple tension and simple compression

During the investigations the longitudinal axis is the *x*, while the cross section's plane is in the *yz* plane. It is also supposed that the investigated material type is linear, isotropic and homogeneous, so the Hooke's law is valid. In the case of simple tension and simple compression there is no change in the shear angles, so the strain state is the following in a solid body point:

$$\boldsymbol{A} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \tag{5.1}$$

where $\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x$. For the stress state the stress tensor can also be determined for a solid body point:

$$\boldsymbol{T} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{5.2}$$

The connection between the stresses and strains can be described by the so called simple Hooke's law:

$$\sigma_x = E\varepsilon_x \tag{5.3}$$

In the case of tension and compression the stress distribution along the area of the cross section is constant and the stress can be calculated by the following equation:

$$\sigma_x = \frac{N}{A}, \tag{5.4}$$

where N is the normal force and A is the area of the cross section. Using the Hooke's law and the above equation the elongation of the investigated beam length can be calculated by the following form:

$$\Delta l = \frac{N \cdot l}{A \cdot E}, \tag{5.5}$$

where *l* is the investigated length.

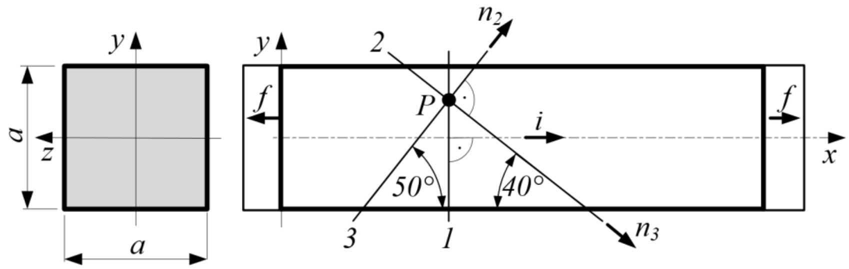
5.2. Examples for tension and compression

Example 1

A prismatic bar with square cross section is loaded by distributed force system with f intensity. The bar is cut across the body point P with three planes (see in figure) denoted with the $\boldsymbol{i}, \boldsymbol{n}_2$ and $\boldsymbol{n}_3$ *[2]*.

Data:

$$f = 250\frac{N}{mm^2}, \qquad a = 100mm$$



Questions:

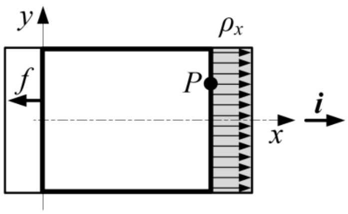
a, Determine the magnitude of the stress vector at body point P (the cutting plane is perpendicular to the bar's axis), then establish the stress tensor of point P!

b, Determine the magnitude of the stress vector at body point P (the normal unit vectors of the investigated planes are $\boldsymbol{n}_2$ and $\boldsymbol{n}_3$). Calculate the magnitude of the coordinates of the $\boldsymbol{\varrho}_{n_i}$ stress vector in the local coordinate system determined by the $\boldsymbol{n}_i, \boldsymbol{m}_i$ and $\boldsymbol{k}_i$ unit vectors!

Solution:

a, In the immediate neighbourhood of body point P the stress vector can be derived by the following form:

$$\boldsymbol{\rho}_{n_i} = \lim_{\Delta A \to 0} \frac{\Delta \boldsymbol{F}}{\Delta A}$$



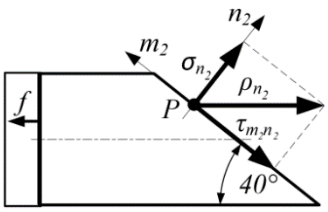
While the stress resultant of the cross sections is distributed force system with constant intensity it can be stated that

$$\boldsymbol{\varrho}_x = \frac{N}{A} \cdot \boldsymbol{i} = \frac{fA}{A} \cdot \boldsymbol{i} = f \cdot \boldsymbol{i} = 250\boldsymbol{i}\frac{N}{mm^2} = 250\boldsymbol{i}MPa$$

In the coordinate system determined by the $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ unit vectors the direction of $\boldsymbol{\varrho}_x$ stress vector is parallel with the $\boldsymbol{i}$ unit vector, therefore $\tau_{yx} = 0$. It results that the stress tensor in the body point P has only one element

$$\boldsymbol{T}_P = \begin{bmatrix} 250 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$

b, The $\boldsymbol{\varrho}_{n_2}$ stress vector appears in the body point P cut by the second plane can be interpreted using the following figure.



$$|\boldsymbol{\rho}_{n_2}| = \frac{N}{A_2} = \frac{fA}{A_2} = f \sin 40° = 160.7MPa$$

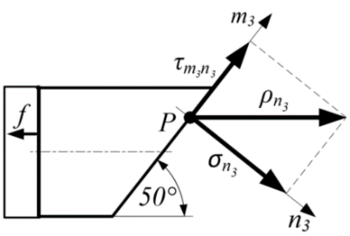
$$A_2 = a\frac{a}{\sin 40°} = \frac{a^2}{\sin 40°} = \frac{A}{\sin 40°}$$

The $\boldsymbol{\varrho}_{n_2}$ stress vector will have components in normal and tangential directions in the coordinate system determined by $\boldsymbol{n}_2, \boldsymbol{m}_2, \boldsymbol{k}_2$ unit vectors, so

$$\sigma_{n_2} = |\boldsymbol{\rho}_{n_2}| \sin 40\,° = 103.3MPa$$

$$\tau_{m_2 n_2} = |\boldsymbol{\rho}_{n_2}| \cos 40\,° = -123.1MPa$$

The $\boldsymbol{\varrho}_{n_3}$ stress vector appears in the body point P cut by the third plane can be interpreted using the following figure.



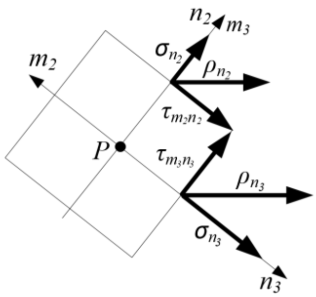
$$|\boldsymbol{\rho}_{n_3}| = \frac{N}{A_3} = \frac{fA}{A_3} = f \sin 50° = 191.51MPa$$

$$A_3 = a\frac{a}{\sin 50°} = \frac{A}{\sin 50°}$$

$$\sigma_{n_3} = \left|\boldsymbol{\rho}_{n_3}\right| \cos 50° = 146.71 MPa$$

$$\tau_{m_2 n_2} = \left|\boldsymbol{\rho}_{n_3}\right| \sin 50° = 123.1 MPa = -\tau_{m_2 n_2}$$

It is worth to observe that magnitude of the shear stresses rising in planes that are perpendicular to each other are the same. The explanation of this phenomenon is coming from the duality of the shear stresses.



The $\boldsymbol{\varrho}_{n_3}$ stress vector and its components can be calculated using the presented process. For this purpose the $\boldsymbol{n}_3$ and $\boldsymbol{m}_3$ unit vectors have to be determined firstly.

$$\boldsymbol{n}_3 = \cos 40° \boldsymbol{i} - \sin 40° \boldsymbol{j}, \boldsymbol{m}_3 = \cos 50° \boldsymbol{i} + \sin 50° \boldsymbol{j}$$

Finally, starting from the $\boldsymbol{T}_P$ stress tensor determined earlier the $\boldsymbol{\rho}_{n_3}$ stress vector and its components can be calculated.

$$\boldsymbol{\rho}_{n_3} = \boldsymbol{T}_P \cdot \boldsymbol{n}_3 = \begin{bmatrix} 250 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos 40° \\ -\sin 40° \\ 0 \end{bmatrix} MPa = \begin{bmatrix} 250 \cdot \cos 40° \\ 0 \\ 0 \end{bmatrix} MPa = \begin{bmatrix} 191.51 \\ 0 \\ 0 \end{bmatrix} MPa$$

$$\sigma_{n_3} = \boldsymbol{\rho}_{n_3} \cdot \boldsymbol{n}_3 = \begin{bmatrix} 191.51 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos 40° \\ -\sin 40° \\ 0 \end{bmatrix} MPa = 146.71 MPa$$

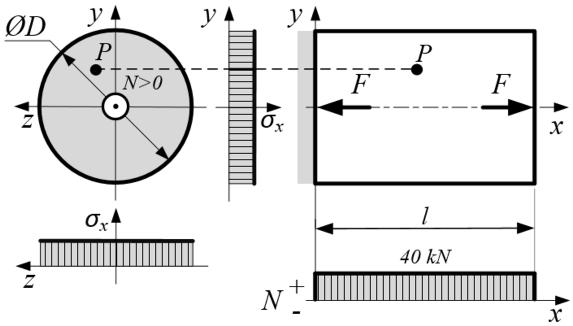
$$\tau_{m_3 n_3} = \boldsymbol{\rho}_{n_3} \cdot \boldsymbol{m}_3 = \begin{bmatrix} 191.51 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos 50° \\ \sin 50° \\ 0 \end{bmatrix} MPa = 123.1 MPa$$

Example 2

Simple loading: tension. A bar with circular cross section (see in figure) is loaded by a force at the bar end.

Data:

$$l = 800\,mm, D = 30\,mm, E = 2.1 \cdot 10^5 MPa, \nu = 0.33, F = 40kN$$



Questions:

a, Draw the normal force diagram of the bar, then determine the dangerous cross section! Determine the magnitude of the internal loading in the dangerous cross section(s)! Draw the stress distribution on the cross section along the axis y and the axis z! Determine the dangerous point(s) of the cross section using the stress distribution!

b, Establish the stress tensor of body point P!

c, Determine the elongation (Δl) of the bar caused by the force and the diameter change (ΔD) in the bar cross section!

e, Establish the strain tensor of body point P using the simple Hooke's law!

Solution:

a, A part of the solution can be seen in the above figure. The dangerous cross sections of the bar are all cross sections.

The magnitude of the internal loading in the dangerous cross section is $N = 40\ kN$

Dangerous points of the cross section are all points of the cross section.

b, The stress tensor in the case of simple loading contains only one element. After substitution the stress tensor at body point P can be calculated. A normal stress can be determined from the normal force and the area of the cross section.

$$\boldsymbol{T}_P = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 56.59 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$

$$\sigma_x(P) = \frac{N}{A} = \frac{N}{\frac{D^2\pi}{4}} = \frac{40\ kN \cdot 10^3}{706.86\ mm^2} = 56.59 MPa$$

c, To be able to determine the elongation of the bar it is needed to calculate the longitudinal normal strain using the simple Hooke's law.

$$\varepsilon_x = \frac{l' - l}{l} = \frac{\Delta l}{l} \rightarrow \Delta l = l \cdot \varepsilon_x = 800\,mm \cdot 2.69 \cdot 10^{-4} = 0.21558 mm$$

$$\sigma_x = E \cdot \varepsilon_x \Rightarrow \varepsilon_x = \frac{\sigma_x}{E} = \frac{56.59 MPa}{2.1 \cdot 10^5 MPa} = 2.69 \cdot 10^{-4}$$

To be able to determine the cross section change of the cross section it is needed to calculate the cross sectional normal strain using the Poisson ratio of the material used.

$$\varepsilon_k = \frac{D' - D}{D} = \frac{\Delta D}{D} \Rightarrow \Delta D = D \cdot \varepsilon_k = 30\,mm \cdot (-0.8089 \cdot 10^{-4}) = -2.668 \cdot 10^{-3} mm$$

$$\varepsilon_k = \varepsilon_y = \varepsilon_z = -\nu \cdot \varepsilon_x = -0.8089 \cdot 10^{-4}$$

d, While in the case of simple tension the shear angles are zero, the strain tensor contains only the normal strains (longitudinal and cross sectional).

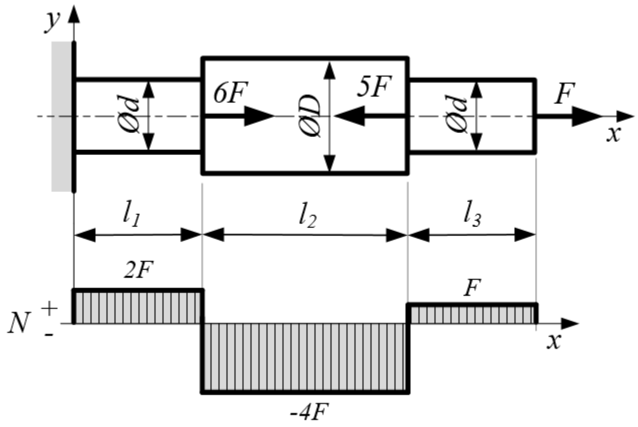
$$\boldsymbol{A}_P = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \begin{bmatrix} 2.69 & 0 & 0 \\ 0 & -0.8089 & 0 \\ 0 & 0 & -0.8089 \end{bmatrix} 10^{-4}$$

Example 3

Simple loading: tension and compression. A bar with changing cross section is loaded by normal forces (see in figure).

Data:

$$d = 100mm, D = 150mm, l_1 = l_3 = 0.2m, l_2 = 0.3m, F = 100kN, E = 0.7 \cdot 10^5\ MPa$$



Questions:

a, Determine the normal force diagram!

b, Determine the normal stresses rising in different sections of the bar!

c, Determine the elongation of some bar sections, then calculate the total elongation of the bar!

Solution:

a, The normal force diagram can be seen in the above figure. The internal loadings are listed for the different bar cross sections:

$$N_1 = 200kN$$

$$N_2 = -400kN$$

$$N_3 = 100kN$$

b, From engineering intuition the section 1 and 2 seem to be dangerous, which can be controlled with calculating the normal stresses:

$$\sigma_1 = \frac{N_1}{A_1} = \frac{200 \cdot 10^3\ N}{7854mm^2} = 24.56MPa$$

$$A_1 = \frac{d^2\pi}{4} = 7854mm^2$$

$$\sigma_2 = \frac{N_2}{A_2} = \frac{-400 \cdot 10^3\ N}{17671mm^2} = -22.64MPa$$

$$A_2 = \frac{D^2\pi}{4} = 17671mm^2$$

From the results it can be seen the dangerous cross sections of the bar are the every cross section of the section 1.

c, The total elongation of the bar can be calculated from the following. It can be seen that the total elongation is the sum of the elongations of the different sections. Using the Equation 5.5 the elongation of the investigated bar sections can be calculated. For these calculations the Young's modulus of the material is needed to be known among others (section length, normal force, area of the cross section).

$$\Delta l = \sum_{i=1}^{3} \Delta l_i = \frac{l_i \cdot N_i}{A_i \cdot E}$$

$$\Delta l_1 = \frac{F_1 \cdot l_1}{E \cdot A_1} + \frac{F_2 \cdot l_2}{E \cdot A_2} + \frac{F_3 \cdot l_3}{E \cdot A_3} = \frac{N_1 \cdot l_1}{E \cdot l_1} + \frac{N_2 \cdot l_2}{E \cdot l_2} + \frac{N_3 \cdot l_3}{E \cdot l_3}$$

$$\Delta l_1 = \frac{l_1 \cdot N_1}{A_1 \cdot E} = \frac{200\ mm \cdot 2 \cdot 10^5\ N}{7854 mm^2 \cdot 0.7 \cdot 10^5\ MPa} = 0.07276\ mm$$

$$\Delta l_2 = \frac{l_2 \cdot N_2}{A_2 \cdot E} = \frac{300\ mm \cdot (-4 \cdot 10^5\ N)}{17671 mm^2 \cdot 0.7 \cdot 10^5\ MPa} = -0.09701\ mm$$

$$\Delta l_3 = \frac{l_3 \cdot N_3}{A_3 \cdot E} = \frac{200\ mm \cdot 10^5\ N}{7854 mm^2 \cdot 0.7 \cdot 10^5\ MPa} = 0.03638\ mm = \frac{\Delta l_1}{2}$$

$$\Delta l = \Delta l_1 + \Delta l_2 + \Delta l_3 = 0.01214 mm$$

From the total elongation it can be seen that the investigated bar elongates.

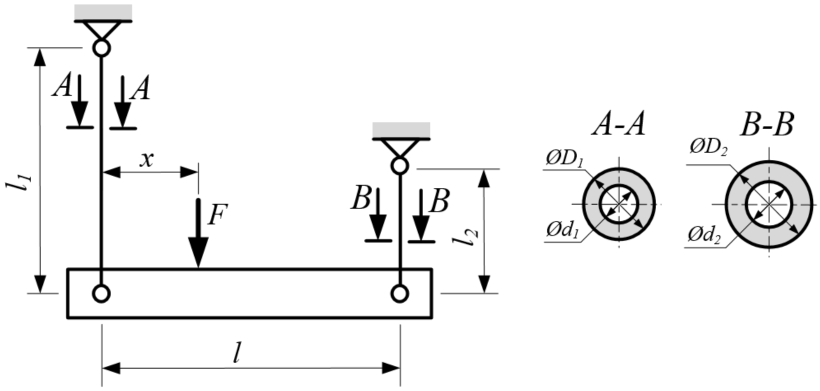
Example 4

Simple loading: tension. Bars (links) produced from different materials and different cross sections support a rigid beam (see in figure).

Data:

$$l = 1.5\ m = 1500 mm, l_1 = 1.5 m = 1500 mm, l_2 = 0.5 m = 500 mm$$

$$d_1 = 20 mm, D_1 = 30 mm, d_2 = 35 mm, D_2 = 40 mm, E_1 = 210 GPa, E_2 = 70 GPa$$



Question:

Determine the location of the concentrate force on the rigid beam that the beam can move only parallel to its original position!

Solution:

The condition is that the elongation of the links are the same, that means

$$\Delta l_1 = \Delta l_2$$

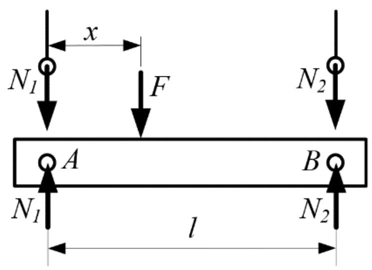
$$\frac{l_1\, N_1}{E_1\, A_1} = \frac{l_2\, N_2}{E_2\, A_2}$$

where the area of the cross sections are

$$A_1 = \frac{(D_1^2 - d_1^2)\cdot\pi}{4} = 393mm^2$$

$$A_2 = \frac{(D_2^2 - d_2^2)\cdot\pi}{4} = 295mm^2$$

The normal forces N_1 and N_2 are unknown, those values are the function of the location of the concentrate force.



To be able to determine the normal forces parametrically the equations of the equilibrium (for this problem the moment equations) is used.

$$\sum M_A = 0 = -F\,x + l\,N_2 \Longrightarrow \;\; N_2 = \frac{F \cdot x}{l}$$

$$\sum M_B = 0 = -N_1\,l + F(l - x) \Longrightarrow N_1 = \frac{F(l - x)}{l}$$

After substituting the normal force into the basic equation (condition mentioned above) the exact location of the concentrate force can be determined as follows:

$$\frac{l_1 \cdot \frac{F(l - x)}{l}}{E_1\,A_1} = \frac{l_2 \cdot \frac{F\,x}{l}}{E_2\,A_2}$$

$$\frac{l_1(l - x)}{E_1\,A_1} = \frac{l_2\,x}{E_2\,A_2}$$

$$\frac{l - x}{x} = \frac{l_2\,E_1\,A_1}{l_1\,E_2\,A_2}$$

$$\frac{1500mm - x}{x} = \frac{500mm \cdot 210 \cdot 10^3 MPa \cdot 393mm^2}{1500mm \cdot 70 \cdot 10^3 MPa \cdot 295mm^2}$$

$$\frac{1500mm - x}{x} = \frac{393}{295} = 1.3322$$

$$1500mm = 2.3322x$$

$$x = 643.17mm$$

6. SIMPLE LOADINGS - BENDING

6.1. Theoretical background of bending

Let us investigate a prismatic beam where the resultant of the stresses in any arbitrary cross section is a bending moment. Our assumptions are that the *y-z* plane is the cross sectional plane and the *y* axis is the symmetry axis of the cross section.

It can be observed through experiments that the length of the neutral axis of the prismatic beam does not change (original length remains the same). It can be also noted the original plane cross section remains plane and perpendicular to the neutral surface which becomes roll shell under loading. It means that similarly to the tension and compression there is no shear angles. From this observation the strain state is the following in a solid body point:

$$\boldsymbol{A} = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} \tag{6.1}$$

where $\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x$. For the stress state the stress tensor can also be determined for a solid body point:

$$\boldsymbol{T} = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6.2}$$

The connection between the stresses and strains can be described by the so called simple Hooke's law:

$$\sigma_x = E\varepsilon_x \tag{6.3}$$

The longitudinal normal strain can be expressed by the following form

$$\varepsilon_x = \frac{y}{R} \tag{6.4}$$

where y is the distance from the neutral chord and R is the curve radius. From measurement we cannot measure the curve radius of the deformed beam but in the stress calculation the curve radius can be included by the following using the simple Hooke's law and from the observation (Equation 6.3 and Equation 6.4)

$$\sigma_x = E\frac{y}{R} \tag{6.5}$$

While the normal stress can't be determined from the laboratory measurement we have to call the equations of the equilibrium. The distributed internal force system has to be equivalent to the stress resultant. The loading is bending moment (M_{bz}).

From the force equation using Equation 6.5

$$\boldsymbol{F}_S = 0 = \int_{(A)} \boldsymbol{\varrho}_x dA = \int_{(A)} \sigma_x \boldsymbol{i} dA = \boldsymbol{i}\frac{E}{R}\int_{(A)} y dA \tag{6.6}$$

There is no external force and the integral $\int_{(A)} ydA$ is the S_x static moment (first moments of the area in *mm*[3]. While the neutral surface goes through the centroid of the cross section the static moment is zero.

From the moment equation using Equation 6.5

$$\begin{aligned} \boldsymbol{M}_S &= -M_{bz}\boldsymbol{k} = \int_{(A)} \boldsymbol{r} \times \boldsymbol{\varrho}_x dA = \int_{(A)} (y\boldsymbol{j} + z\boldsymbol{k}) \times \boldsymbol{\varrho}_x dA = \\ &= \int_{(A)} (y\boldsymbol{j} + z\boldsymbol{k}) \times \sigma_x \boldsymbol{i} dA = -\int_{(A)} \sigma_x y \boldsymbol{k} dA + \int_{(A)} \sigma_x z \boldsymbol{j} dA = \\ &= -\frac{E}{R}\boldsymbol{k}\int_{(A)} y^2 dA + \frac{E}{R}\boldsymbol{j}\int_{(A)} yz dA \end{aligned} \tag{6.7}$$

The integral $\int_{(A)} y^2 dA$ is the I_z moment of inertia (in detailed the area moment of inertia respect to the axis z and $I_z > 0$ in *mm*[4]), and the integral $\int_{(A)} yzdA$ is the I_{yz} product of inertia (the $I_{yz} = I_{zy}$ and equal to zero in this case). It means that the following equation can be expressed from Equation 6.7

$$-M_{bz}\boldsymbol{k} = -\frac{E}{R} \cdot I_z \cdot \boldsymbol{k} \tag{6.8}$$

Reordering Equation 6.8 we get

$$\frac{M_{bz}}{I_z E} = \frac{1}{R} \tag{6.9}$$

Finally using Equation 6.5 the normal stress for bending can be introduced with the following equation

$$\sigma_x = \frac{M_{bz}}{I_z} \cdot y \tag{6.10}$$

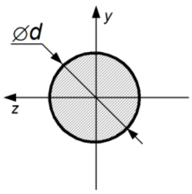
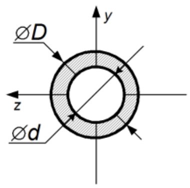
It has to be noted that we are talking about only bending when shear force is in the system. In this case we neglect the stress and strain come from the shear force, because it is to small compared to the stress and strain come from the bending moment.

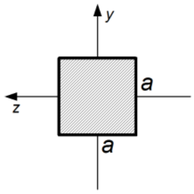
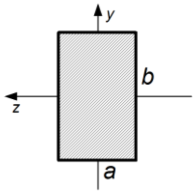
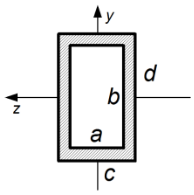
The maximum normal stress for bending can be calculated with the following way

$$\sigma_{xmax} = \frac{M_{bz}}{I_z} \cdot y_{max} = \frac{M_{bz}}{K_z} \qquad (6.11)$$

where K_z is the section modulus.

Below the most important cross sectional properties are listed of simple cross sections:

Cross section	 circular	 tube (ring)
$A\ [mm^2]$ area of the cross section	$\frac{d^2\pi}{4}$	$\frac{(D^2-d^2)\pi}{4}$
$I_y\ [mm^4]$ moment of inertia of the cross section respect to axis y	$\frac{d^4\pi}{64}$	$\frac{(D^4-d^4)\pi}{64}$
$I_z\ [mm^4]$ moment of inertia of the cross section respect to axis z	$\frac{d^4\pi}{64}$	$\frac{(D^4-d^4)\pi}{64}$
$K_y[mm^3]$ section modulus respect to axis y	$\frac{d^3\pi}{32}$	$\frac{(D^4-d^4)\pi}{32D}$
$K_z[mm^3]$ section modulus respect to axis z	$\frac{d^3\pi}{32}$	$\frac{(D^4-d^4)\pi}{32D}$

Cross section	 square	 rectangle
$A\ [mm^2]$ area of the cross section	a^2	ab
$I_y\ [mm^4]$ moment of inertia of the cross section respect to axis y	$\frac{a^4}{12}$	$\frac{a^3b}{12}$
$I_z\ [mm^4]$ moment of inertia of the cross section respect to axis z	$\frac{a^4}{12}$	$\frac{ab^3}{12}$
$K_y[mm^3]$ section modulus respect to axis y	$\frac{a^3}{6}$	$\frac{a^2b}{6}$
$K_z[mm^3]$ section modulus respect to axis z	$\frac{a^3}{6}$	$\frac{ab^2}{6}$
Cross section	 rectangular tube	
$A\ [mm^2]$ area of the cross section	$cd - ab$	
$I_y\ [mm^4]$ moment of inertia of the cross section respect to axis y	$\frac{c^3d}{12} - \frac{a^3b}{12}$	

I_z $[mm^4]$ moment of inertia of the cross section respect to axis z	$\frac{cd^3}{12}-\frac{ab^3}{12}$
$K_y[mm^3]$ section modulus respect to axis y	$\frac{2I_y}{c}$
$K_z[mm^3]$ section modulus respect to axis z	$\frac{2I_z}{d}$

6.2. Examples for bending of beams

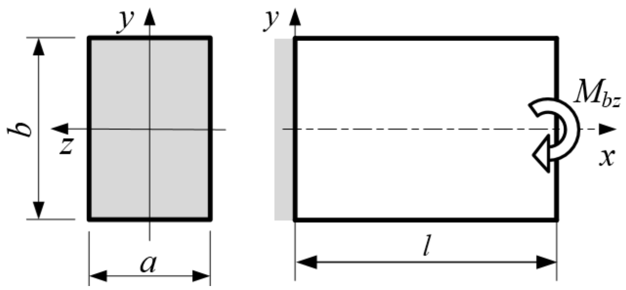
Example 1

Simple loading: bending. A prismatic beam with rectangle cross section is loaded (see in figure)!

Data:

$$R_{p\,0.2}=320MPa, n=1.6, P=\left(\frac{l}{2};-20;-10\right)mm, a=40mm, b=60mm$$

$$M_{bz}=4kNm, E=2.1\cdot 10^5 MPa, \nu=0.3$$



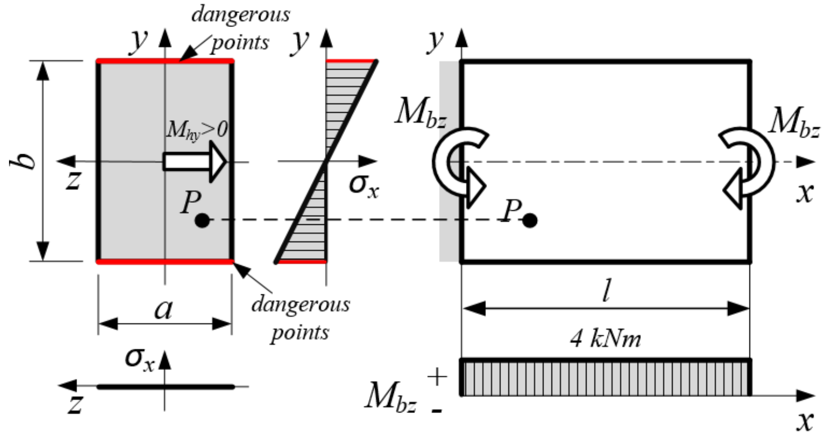
Questions:

a, Determine the stress and strain tensor of solid body point P!

b, Control the beam for stress!

Solution:

a, To be able to calculate the stress and strain components the bending moment diagram, the dangerous cross section, stress distribution and the dangerous points have to be determined (see in figure below).



After calculating the moment of inertia of the cross section the normal stress at body point *P* can be determined.

$$\sigma_x(P) = \frac{M_{bz}}{I_z} y_P = \frac{4\ kNm}{7.2 \cdot 10^5 mm^4} \cdot (-20mm) = \frac{4 \cdot 10^6 Nmm}{7.2 \cdot 10^5 mm^4} \cdot (-20mm)$$
$$= -111.1 \frac{N}{mm^2} = -111.1MPa$$

$$I_z = \frac{ab^3}{12} = \frac{40 \cdot 60^3}{12} = 720000mm^4$$

In general the stress tensor for bending moment is the following and after substitution the stress state can be calculated for body point *P*.

$$\boldsymbol{T}_P = \begin{bmatrix} \sigma_x(P) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -111.\dot{1} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$

For the determination of the strain state of body point *P* the simple Hooke's law has to used to be able calculate the normal strains as follows

$$\varepsilon_x = \frac{\sigma_x(P)}{E} = \frac{-111.\dot{1}MPa}{2{,}1 \cdot 10^5} = -5.291 \cdot 10^{-4}$$

$$\varepsilon_y = \varepsilon_z = -\nu\varepsilon_x = -0.3 \cdot (-5.291 \cdot 10^{-4}) = 1.587 \cdot 10^{-4}$$

In general the stress tensor for bending moment is the following and after substitution the stress state can be calculated for body point *P*.

$$\boldsymbol{A}_P = \begin{bmatrix} \varepsilon_x & 0 & 0 \\ 0 & \varepsilon_y & 0 \\ 0 & 0 & \varepsilon_z \end{bmatrix} = \begin{bmatrix} -5.291 & 0 & 0 \\ 0 & 1.587 & 0 \\ 0 & 0 & 1.587 \end{bmatrix} \cdot 10^{-4}$$

b, To be able to control the for the maximum stress, the basic equation of sizing and control of beam structures has to be written.

$$\sigma_{xmax} \leq \sigma_{all}$$

where the allowable stress for the material can be calculated from the yield point and the factor of safety.

$$\sigma_{meg} = \frac{R_{p\,0.2}}{n} = \frac{320MPa}{1.6} = 200MPa$$

The maximum normal stress can be calculated using the bending moment and the section modulus.

$$\sigma_{x\,max} = \frac{M_{bz}}{I_z} y_{max} = \frac{M_{bz}}{K_z} = \frac{4 \cdot 10^6}{2.4 \cdot 10^4} = 166.67MPa$$

where the section modulus of the rectangular cross section is

$$K_z = I_z \frac{1}{y_{max}} = \frac{ab^3}{12} \cdot \frac{2}{b} = \frac{ab^2}{6} = \frac{40 \cdot 60^2}{6} = 2.4 \cdot 10^4 mm^3$$

Let us examine if the basic equation of sizing and control is fulfilled or not.

$$\sigma_{x\,max} \leq \sigma_{all} \quad \rightarrow 166.67MPa \leq 200MPa$$

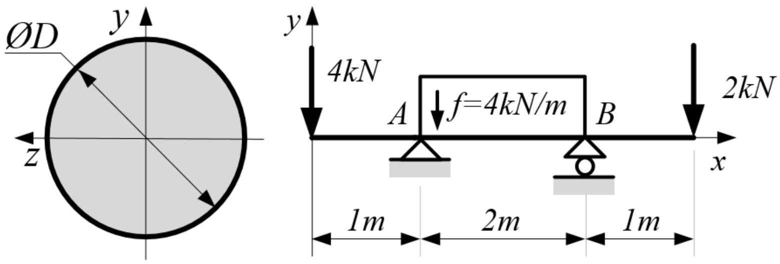
While the equation for control is fulfilled the beam is met with the requirement in the point of stress view.

Example 2

Simple loading: bending. A prismatic beam with circular cross section is loaded (see in figure)!

Data:

$$\sigma_{meg} = 170MPa$$

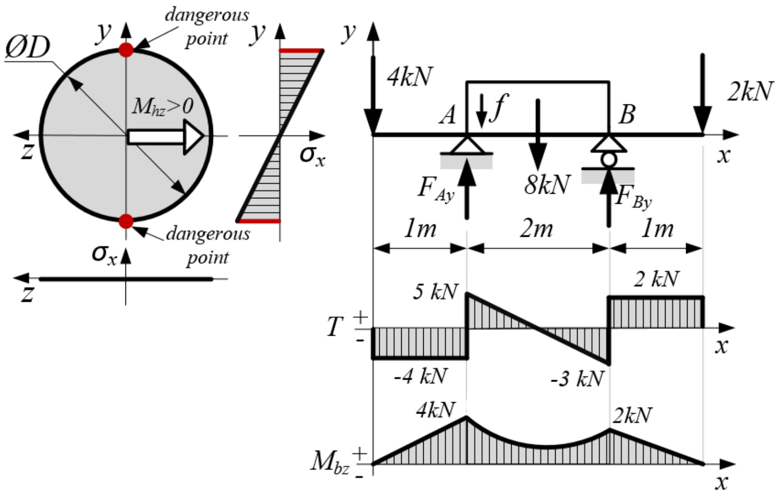


Question:

Size the beam for maximum stress (during the investigation we neglect stress and strain come from shear)!

Solution:

To be able to size the beam the shear force diagram, the bending moment diagram, the dangerous cross section, stress distribution and the dangerous points have to be determined (see in figure below).



To be able to size the beam for the maximum stress, the basic equation of sizing and control of beam structures has to be written.

$$\sigma_{xmax} \leq \sigma_{all}$$

For this example the following is valid.

$$\sigma_{xmax} = \frac{|M_{bz}|}{K_z} \leq \sigma_{all} \Longrightarrow K_z \geq \frac{|M_{bz}|}{\sigma_{meg}}$$

After reordering the minimum section modulus needs to be calculated.

$$K_{zmin} = \frac{|M_{bz}|}{\sigma_{meg}} = \frac{|4 \cdot 10^6 Nmm|}{170MPa} = 2.353 \cdot 10^4 \; mm^3$$

Using the minimum section modulus for circular cross section the minimum required diameter can be determined.

$$K_{zmin} = \frac{D^3 \cdot \pi}{32} \Longrightarrow D_{min} = \sqrt[3]{\frac{32 \cdot K_{zmin}}{\pi}} = \sqrt[3]{\frac{32 \cdot 2.353 \cdot 10^4 mm^3}{\pi}} = 62.11mm$$

After rounding up the minimum required diameter the applied diameter can be determined.

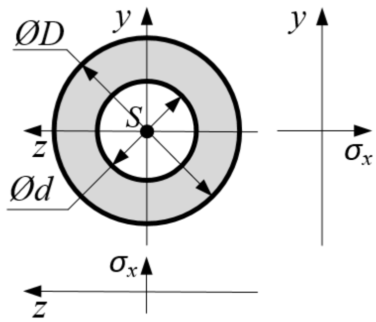
$$D_{app} = 65mm$$

Example 3

Simple loading: bending. A prismatic beam with ring cross section is loaded. The loading is given with the moment vector reduced to the centre point of the cross section. The diameter ration is also given.

Data:

$$\sigma_{all} = 100MPa, \qquad \frac{D}{d} = \frac{4}{3}, \qquad \boldsymbol{M}_S = 4.5\boldsymbol{k}\ kNm$$

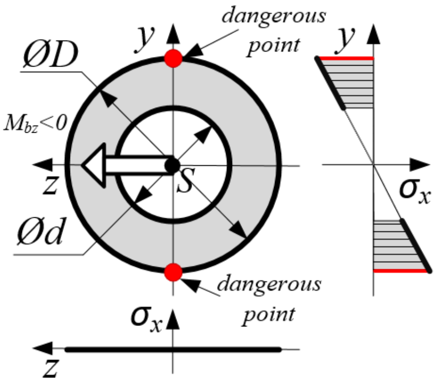


Question:

Size the beam for the maximum stress (during the investigation we neglect stress and strain come from shear)!

Solution:

To be able to size the beam the stress distribution and the dangerous points have to be determined on the cross section (see in figure below).



To be able to size the beam for the maximum stress, the basic equation of sizing and control of beam structures has to be written.

$$\sigma_{xmax} \leq \sigma_{all}$$

For this example the following is valid.

$$\sigma_{xmax} = \frac{|M_{bz}|}{K_z} \leq \sigma_{meg} \rightarrow K_z \geq \frac{|M_{hz}|}{\sigma_{meg}}$$

After reordering the minimum section modulus needs to be calculated.

$$K_{zmin} = \frac{|M_{bz}|}{\sigma_{meg}} = \frac{|-4{,}5 \cdot 10^6 Nmm|}{100MPa} = 4.5 \cdot 10^4 mm^3$$

Using the minimum section modulus for ring cross section the minimum required diameter can be determined.

$$K_z = \frac{(D^4 - d^4)\pi}{32D} = \frac{\left[\left(\frac{4}{3}d\right)^4 - d^4\right]\pi}{32 \cdot \frac{4}{3}d} = \frac{\left[\left(\frac{256}{81}d^4\right) - d^4\right]\pi}{\frac{128}{3}d} = \frac{\frac{175}{81}d^4\pi}{\frac{128}{3}d} = \frac{175d^3\pi}{27 \cdot 128}$$

$$d_{min} = \sqrt[3]{\frac{27 \cdot 128 \cdot K_{zmin}}{175\pi}} = 65.64\ mm$$

After rounding up the minimum required diameter the applied diameters (inner and outer diameters) can be determined.

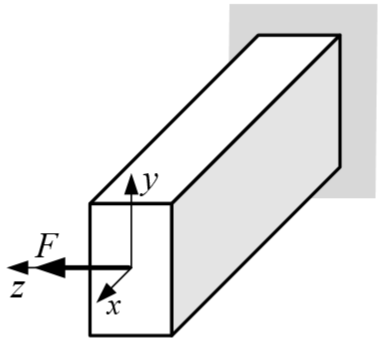
$$d_{app} = 66mm, D_{app} = 88mm$$

Example 4

Simple loading: bending. A prismatic beam with rectangle cross section is loaded (see in figure)!

Data:

$$R_P = 450MPa, \qquad n = 2.5, \quad l = 2m$$



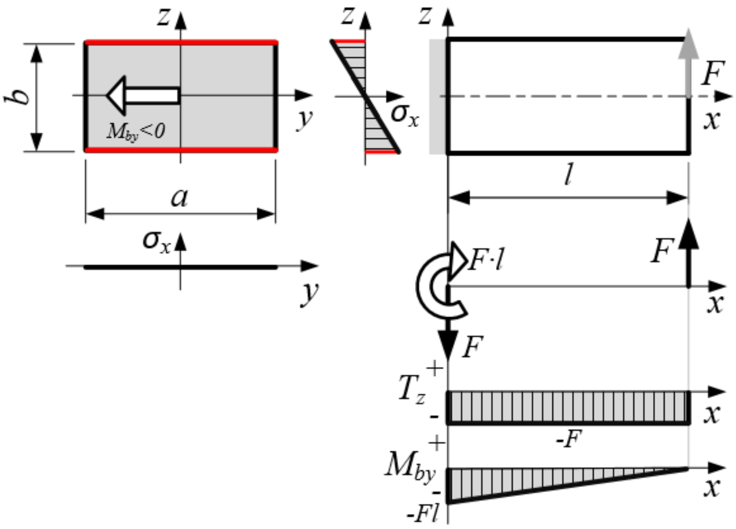
Questions:

a, Size the beam for the maximum applied load (during the investigation we neglect stress and strain come from shear)!

b, Size the beam for the load acting along the axis y (during the investigation we neglect stress and strain come from shear)!

Solution:

a, To be able to size the beam the shear force diagram, the bending moment diagram, the dangerous cross section, stress distribution and the dangerous points have to be determined (see in figure below).



To be able to size the beam for the maximum stress (then for maximum applied load), the basic equation of sizing and control of beam structures has to be written.

$$\sigma_{xmax} \le \sigma_{all}$$

For this example the following is valid.

$$\sigma_{xmax} = \frac{|M_{by}|}{K_y} \leq \sigma_{all} = \frac{R_P}{n} = 180MPa \rightarrow M_{by} \leq \sigma_{all} K_y$$

After reordering the maximum applied moment can be calculated.

$$M_{bymax} = \sigma_{all} K_y = 180MPa \cdot 2000\, mm^3 = 360000Nmm$$

where the section modulus respect to the axis y is

$$K_y = \frac{ab^2}{6} = 2000mm^3$$

While the question was the determination of the maximum applied load, form the maximum bending moment this value can easily be calculated as follows:

$$M_{by} = F \cdot l \rightarrow F_{max} = \frac{M_{by}}{l} = \frac{360000Nmm}{2000mm} = 180N$$

b, To be able to size the beam for the maximum stress (then for the load acting along the axis y), the basic equation of sizing and control of beam structures has to be written.

$$\sigma_{xmax} \leq \sigma_{all}$$

For this example the following is valid.

$$\sigma_{x\, max} = \frac{|M_{bz}|}{K_z} \leq \sigma_{all} \rightarrow M_{hz} \leq \sigma_{meg} K_z$$

$$M_{bzmax} = \sigma_{all} K_z = 180MPa \cdot 3000\, mm^3 = 540000Nmm$$

where the section modulus respect to the axis z is

$$K_z = \frac{a^2 b}{6} = 3000mm^3$$

While the question was the determination of the maximum applied load acting along the axis y, form the maximum bending moment this value can easily be calculated as follows:

$$M_{bz} = F \cdot l \rightarrow F_{max} = \frac{M_{by}}{l} = \frac{540000Nmm}{2000mm} = 270N$$

7. SIMPLE LOADINGS - TORSION

7.1. Theoretical background of torsion

In this chapter we will only investigate prismatic beams with circular or ring cross section for pure torsion. The resultant of the stresses in any arbitrary cross section is torsion. It can be stated from a simple torsion test that the circular cross sections of the beam remain circular during the torsion and their diameters and distances between them do not change. For the investigation the cylindrical coordinate system (R, φ, x) is used. The loading is a torque moment (M_t).

The state of strains described by the strain tensor is

$$\boldsymbol{A}\ (R, \varphi, x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2}\gamma_{\varphi x} \\ 0 & \frac{1}{2}\gamma_{x\varphi} & 0 \end{bmatrix} \tag{7.1}$$

While the diameter is not changing during torsion the $\varepsilon_R = 0$. The original length is not changing as well so the $\varepsilon_x = 0$. The state of stresses described by the stress tensor is

$$\boldsymbol{T}\ (R, \varphi, x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau_{\varphi x} \\ 0 & \tau_{x\varphi} & 0 \end{bmatrix} \tag{7.2}$$

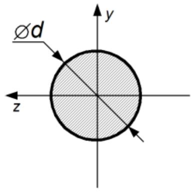
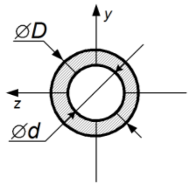
where $\tau_{\varphi x} = \tau_{x\varphi}$ is the shear stress in the cross section. While the normal strains are zero the normal stresses are zero as well. The distribution of the shear stress along the cross section is linear and in the centre of the cross section is zero. The maximum shear stresses arise in the outside fibers. In the cylindrical coordinate system the shear stress can be introduced by the following way,

$$\tau_{\varphi x} = \frac{M_t}{I_p} \cdot \varrho \tag{7.3}$$

where ϱ is the radius to the point of interest and I_p is the polar moment of inertia. By definition the polar moment of inertia is

$$I_p = \int_{(A)} \varrho^2\, dA \tag{7.4}$$

Below the cross sectional properties of circular and ring cross sections are listed:

Cross section	 circular	 tube (ring)
I_p $[mm^4]$ polar moment of inertia of the cross section	$\frac{d^4\pi}{32}$	$\frac{(D^4 - d^4)\pi}{32}$
$K_p[mm^3]$ polar section modulus of the cross section	$\frac{d^3\pi}{16}$	$\frac{(D^4 - d^4)\pi}{16D}$

For pure torsion of linear, elastic and homogenous materials the Hooke's law is valid, therefore the connection between the stress and strain can be described by

$$\tau_{\varphi x} = G \cdot \gamma_{\varphi x} \tag{7.5}$$

While the dangerous points of the cross section are the points of the outer diameter the maximum shear stress can be expressed as follows,

$$\tau_{\varphi x max} = \frac{|M_t|}{I_p} \cdot \varrho_{max} = \frac{|M_t|}{K_p} \tag{7.6}$$

In the case of torsion another important quantity is the angular displacement (angle of twist)

$$\phi = \frac{M_t \cdot l}{I_p \cdot G} \tag{7.7}$$

where l is the investigated length of the beam.

After transformation of the problem into the xyz coordinate system we get following for the shear stress components

$$\tau_{xy} = \tau_{yx} = -\frac{M_t}{I_p} \cdot z_P, \quad \tau_{zx} = \tau_{xz} = \frac{M_t}{I_p} \cdot y_P \tag{7.8}$$

where y_P is the distance of the investigated point from the z axis and z_P is the distance of the investigated point from the y axis.

In the *xyz* coordinate system the stress tensor is

$$\boldsymbol{T}\,(x, y, z) = \begin{bmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & 0 & 0 \\ \tau_{zx} & 0 & 0 \end{bmatrix} \tag{7.9}$$

so the normal stresses are zero ($\sigma_x = \sigma_y = \sigma_z = 0$).

In the *xyz* coordinate system the strain tensor is

$$\boldsymbol{A}\,(x, y, z) = \begin{bmatrix} 0 & \frac{1}{2}\gamma_{xy} & \frac{1}{2}\gamma_{xz} \\ \frac{1}{2}\gamma_{yx} & 0 & 0 \\ \frac{1}{2}\gamma_{zx} & 0 & 0 \end{bmatrix} \tag{7.10}$$

so the normal strains are zero ($\varepsilon_x = \varepsilon_y = \varepsilon_z = 0$).

Using the simple Hooke's law the shear angles can be introduced in the *xyz* coordinate system as

$$\tau_{xy} = \tau_{yx} = G \cdot \gamma_{yx} = G \cdot \gamma_{xy}, \quad \tau_{zx} = \tau_{xz} = G \cdot \gamma_{zx} = G \cdot \gamma_{xz} \tag{7.11}$$

Using Equation 7.6 and the maximum shear stress in the *xyz* coordinate system is

$$\tau_{max} = \frac{|M_t|}{K_p} \tag{7.12}$$

The basic equation of sizing and control for pure torsion is

$$\tau_{max} \leq \tau_{all} \tag{7.13}$$

where τ_{all} is the allowable stress for torsion.

The energy of strain for torsion can be determined as follows (if the M_t and I_p are constant)

$$U_T = \frac{1}{2} \cdot \frac{M_t^2 \cdot l}{I_p \cdot G} \tag{7.14}$$

7.2. Examples for torsion

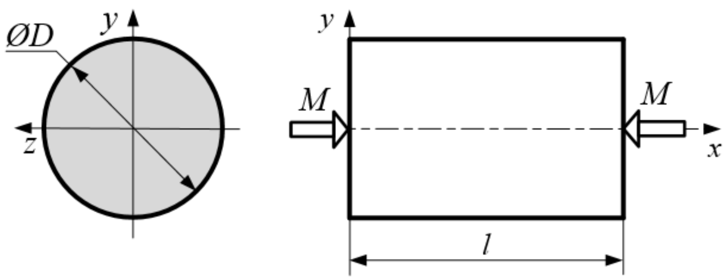
Example 1

Simple loading: pure torsion. A prismatic beam with circular cross section is loaded by torque moment (see in figure). The investigated solid body points (A, B and C) are given with those coordinates.

Data:

$$M = 0.3kNm, \qquad D = 20mm,$$

$$A(0;\ 0; 5)mm, \quad B(0;\ -10; 0)mm, \quad C\left(0;\ -5; -10\frac{\sqrt{3}}{2}\right)mm$$



Questions:

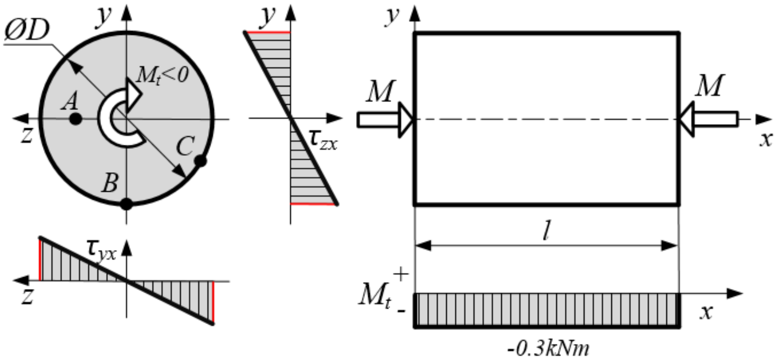
a, Determine the stress tensor of body points A and B!

b, Determine the stress tensor of body point C! Calculate the magnitude of the shear stress at body point C!

c, Determine the stress tensor of body point C in the $R\varphi x$ coordinate system!

Solution:

To be able to calculate the stress components the torque moment diagram and the stress distribution have to be determined (see in figure below). It has to be noted that considering the sign convention of the torsion the torque moment is negative.



a, The stress tensor in general in the case of pure tension in the *xyz* coordinate system is

$$\boldsymbol{T} = \begin{bmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & 0 & 0 \\ \tau_{zx} & 0 & 0 \end{bmatrix}$$

From the figure above it can be read that in point A the shear stress components $\tau_{zx} = \tau_{xz}$ are zero. After calculating the polar moment of inertia of the cross section the shear stress component can be determined.

$$I_p = \frac{D^4 \cdot \pi}{32} = \frac{20^4 \cdot \pi}{32} = 15708mm^4$$

$$\tau_{zx}(A) = \frac{M_t}{I_p} \cdot y_A = 0MPa$$

$$\tau_{xy}(A) = -\frac{M_t}{I_p} \cdot z_A = -\frac{-0.3 \cdot 10^6 Nmm}{15708mm^4} \cdot 5mm = 95.49MPa$$

Substituting the values the stress tensor of body point A can be written as

$$\boldsymbol{T}_A = \begin{bmatrix} 0 & 95.49 & 0 \\ 95.49 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$

From the figure above it can be read that in point B the shear stress components $\tau_{xy} = \tau_{yx}$ are zero. The shear stress component can be determined.

$$\tau_{xy}(B) = -\frac{M_t}{I_p} \cdot z_B = 0MPa$$

$$\tau_{zx}(B) = \frac{M_t}{I_p} \cdot y_B = \frac{-0.3 \cdot 10^6 Nmm}{15708mm^4} \cdot (-10mm) = 190.99MPa$$

Substituting the values the stress tensor of body point B can be written as

$$\boldsymbol{T}_B = \begin{bmatrix} 0 & 0 & 190.99 \\ 0 & 0 & 0 \\ 190.99 & 0 & 0 \end{bmatrix} MPa$$

b, The shear stress components in the body point C can be determined as follows

$$\tau_{xy}(C) = -\frac{M_t}{I_p} \cdot z_C = -\frac{-0.3 \cdot 10^6 Nmm}{15708mm^4} \cdot \left(-10\frac{\sqrt{3}}{2}\right) mm = -165.4MPa$$

$$\tau_{zx}(C) = \frac{M_t}{I_p} \cdot y_C = \frac{-0.3 \cdot 10^6 Nmm}{15708mm^4} \cdot (-5)mm = 95.49MPa$$

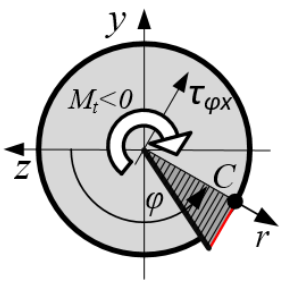
The stress tensor in the body point C is

$$\boldsymbol{T}_C = \begin{bmatrix} 0 & -165.4 & 95.49 \\ -165.4 & 0 & 0 \\ 95.49 & 0 & 0 \end{bmatrix} MPa$$

The magnitude of the shear stress in the body point C is

$$\tau(C) = \sqrt{\left[\tau_{yx}(C)\right]^2 + [\tau_{zx}(C)]^2} = 190.99MPa$$

c, To be able to calculate the stress tensor in body point C the stress distribution has to be drawn (see in figure).



The shear stress can be calculated as

$$\tau_{\varphi x}(C) = \frac{M_t}{I_p} \cdot \varrho_C = \frac{-0.3 \cdot 10^6 Nmm}{15708mm^4} \cdot 10mm = -190.99MPa$$

where $\varrho_C = \frac{D}{2}$. The stress tensor can be established.

$$\boldsymbol{T}_C(R, \varphi, x) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \tau_{\varphi x} \\ 0 & \tau_{x\varphi} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -190.99 \\ 0 & -190.99 & 0 \end{bmatrix} MPa$$

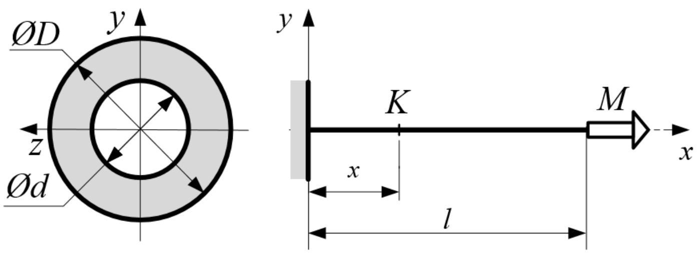
Example 2

Simple loading: pure torsion. A prismatic beam with ring cross section is loaded by torque moment (see in figure). The investigated solid body point P is given with its coordinates *[3].*

Data:

$M_t = 150Nm,\ \ D = 30mm,\ \ d = 24mm,\ \ l = 300mm,\ \ x = 100mm,\ \ G = 80GPa$

$$P(150;\ 0; -12)\ mm$$



Questions:

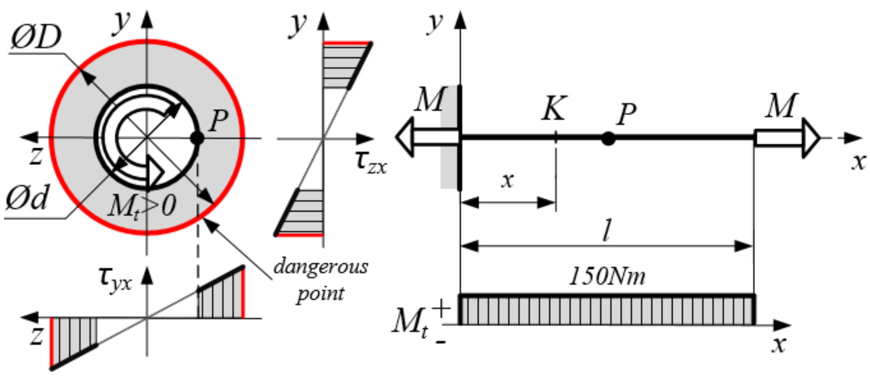
a, Determine the strain and stress tensors in the body point P!

b, Determine the angular displacement of the cross section K!

c, Determine the strain energy of the whole beam and the beam section with length x!

Solution:

To be able to calculate the stress and strain components the torque moment diagram and the stress distribution have to be determined (see in figure below). It has to be noted that considering the sign convention of the torsion the torque moment is positive.



a, After calculating the polar moment of inertia of the cross section the shear stress component can be determined.

$$I_p = \frac{(D^4 - d^4) \cdot \pi}{32} = \frac{(30^4 - 24^4) \cdot \pi}{32} = 46950 \, mm^4$$

$$\tau_{xy}(P) = -\frac{M_t}{I_p} \cdot z_P = -\frac{150 \cdot 10^3 \, Nmm}{46950 \, mm^4} \cdot (-12 \, mm) = 38.34 \, MPa$$

$$\tau_{zx}(P) = \frac{M_t}{I_p} \cdot y_P = 0 \, MPa$$

The stress tensor in general and after substitution

$$\boldsymbol{T}_P = \begin{bmatrix} 0 & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & 0 & 0 \\ \tau_{zx} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 38.34 & 0 \\ 38.34 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$

Using the simple Hooke's law the related shear angle component can be determined.

$$\gamma_{yx}(P) = \frac{\tau_{yx}(P)}{G} = \frac{38.34 \, MPa}{80 \cdot 10^3 \, MPa} = 4.79 \cdot 10^{-4}$$

The strain tensor in general and after substitution

$$\boldsymbol{A}_P = \begin{bmatrix} 0 & \frac{1}{2}\gamma_{xy} & 0 \\ \frac{1}{2}\gamma_{yx} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2.4 & 0 \\ 2.4 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} 10^{-4}$$

b, The angular displacement at cross section K is

$$\phi_K = \frac{M_c \cdot x}{I_p \cdot G} = \frac{150 \cdot 10^3 \, Nmm \cdot 100 \, mm}{46950 \, mm^4 \cdot 80 \cdot 10^3 \, MPa} = 3.99 \cdot 10^{-3} \, rad$$

c, The strain energy of the whole beam is

$$U_T = \frac{1}{2} \cdot \frac{{M_c}^2 \cdot l}{I_p \cdot G} = \frac{1}{2} \cdot \frac{(150 \cdot 10^3 \, Nmm)^2 \cdot 300 \, mm}{46950 \, mm^4 \cdot 80 \cdot 10^3 \, MPa} = 898.56 \, Nmm = 0.899 \, J$$

while the strain energy at the beam section with length x is

$$U_T(K) = \frac{1}{2} \cdot \frac{{M_c}^2 \cdot x}{I_p \cdot G} = \frac{1}{2} \cdot \frac{(150 \cdot 10^3 \, Nmm)^2 \cdot 100 \, mm}{46950 \, mm^4 \cdot 80 \cdot 10^3 \, MPa} = 299.52 \, Nmm = 0.29952 \, J$$

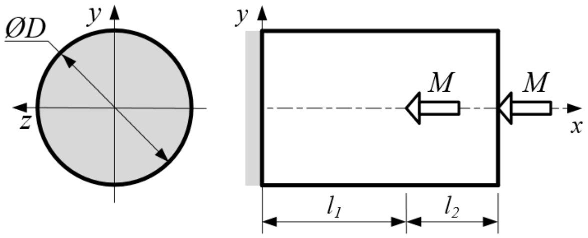
Example 3

Simple loading: pure torsion. A prismatic beam with circular cross section is loaded with torque moments (see in figure). The allowable shear stress and the allowable angular displacement are also known.

Data:

$$M_t = 250\ Nm,\ \ l_1 = 300\ mm,\ \ l_2 = 200\ mm$$

$$G = 65GPA,\ \ \phi_{all} = 1.5°,\ \ \tau_{all} = 40MPa$$



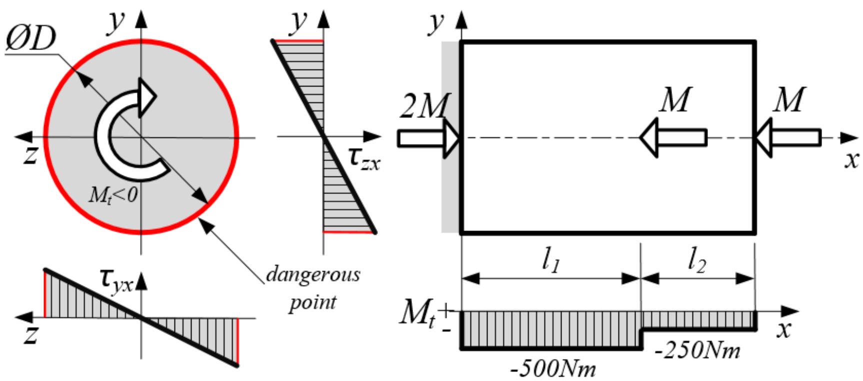
Questions:

a, Size the beam according to the allowable angular displacement for the whole beam!

b, Control the beam for maximum stress according to the allowable shear stress!

Solution:

To be able to size and control the beam, the torque moment diagram, the dangerous cross section, the stress distribution and the dangerous points have to be determined (see in figure below).



a, To be able to size the beam for the allowable angular displacement, the basic equation of sizing and control for angular displacement of beam structures in the case of pure torsion has to be written.

$$\phi_{max} \leq \phi_{all}$$

For this example the following is valid.

$$\phi_{max} = \left|\sum_{i=1}^{2} \frac{M_{t_i} \cdot l_i}{I_p \cdot G}\right| = \frac{1}{I_p \cdot G}\left|\sum_{i=1}^{2} M_{t_i} \cdot l_i\right| \leq \phi_{all} = 1.5 \cdot \frac{\pi}{180°} rad$$

$$I_{p_{min}} = \frac{1}{\phi_{meg} \cdot G}\left|\sum_{i=1}^{2} M_{t_i} \cdot l_i\right| =$$

$$= \frac{|-150 \cdot 10^6\, Nmm^2 - 50 \cdot 10^6\, Nmm^2|}{1.5 \cdot \frac{\pi}{180°} \cdot 65 \cdot 10^3\, MPa} = 117530\, mm^4$$

Using the minimum required polar moment of inertia the minimum required diameter can be determined.

$$I_p = \frac{D^4 \cdot \pi}{32} \rightarrow D_{min} = \sqrt[4]{\frac{32 \cdot I_{pmin}}{\pi}} = 33.08\, mm$$

After rounding up the minimum required diameter we can get the applied diameter.

$$D_{app} = 35\, mm$$

This value for the diameter is suitable for the prescribed allowable angular displacement.

b, To be able to control the beam for the maximum shear stress, the basic equation of sizing and control of beam structures in the case of pure torsion has to be written.

$$\tau_{max} \leq \tau_{all}$$

The applied diameter is determined before so the applied polar section modulus can be determined.

$$K_{papp} = \frac{D_{app}^3 \cdot \pi}{16} = \frac{35^3\, mm^3 \cdot \pi}{16} = 8418\, mm^3$$

Using the applied polar section modulus the maximum shear stress can be calculated.

$$\tau_{max} = \frac{|M_t|}{K_{papp}} = \frac{|-0.5 \cdot 10^6\, Nmm|}{8418\, mm^3} = 59.39\, MPa \leq \tau_{meg}$$

The calculated maximum stress and the allowable has to compared if the condition is fulfilled or not.

$$59.39\, MPa \leq 40\, MPa$$

This result means that the calculated applied diameter is not enough to bear this load. It is needed to size the beam for the maximum stress!

$$\tau_{max} = \frac{|M_t|}{K_p} \leq \tau_{all}$$

The minimum required polar section modulus can be calculated.

$$K_{pmin} = \frac{|M_t|}{\tau_{all}} = \frac{|-0.5 \cdot 10^6\ Nmm|}{40\ MPa} = 12500\ mm^3$$

Using the minimum required polar section modulus the minimum required diameter can be determined.

$$K_{pmin} = \frac{D^3 \cdot \pi}{16} \rightarrow D_{min} = \sqrt[3]{\frac{16 \cdot K_{pmin}}{\pi}} = \sqrt[3]{\frac{16 \cdot 12500\ mm^3}{\pi}} = 39.93\ mm$$

After rounding up the minimum required diameter the applied diameter can be calculated.

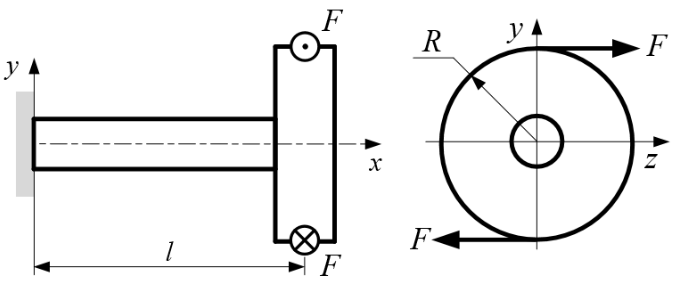
$$D_{alk} = 40\ mm$$

Example 4

Simple loading: pure torsion. A prismatic circular shaft assembled with a disc is loaded. On the circumference of the disc forces are acting.

Data:

$$F = 1kN,\ \ R = 0.15m,\ \ \tau_{all} = 60MPa$$



Questions:

a, Size the circular shaft for maximum stress!

b, Size the shaft for maximum stress if we plan to use ring cross section and the diameter ratio is known ($\frac{D_2}{d_2} = \frac{9}{8}$)!

c, If we plan to reduce the weight of the shaft which cross section would be chosen for production. How much easier it would be in percentage?

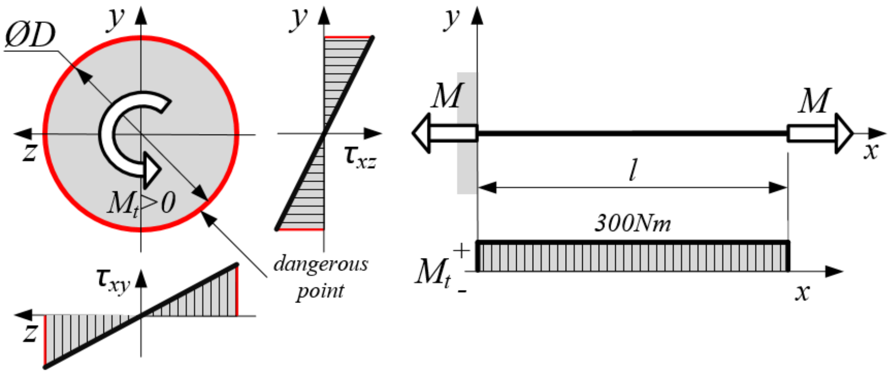
Solution:

The torque moment has to be determined first which comes from the loaded disc.

$$\boldsymbol{F}_S = \sum \boldsymbol{F}_i = -F \cdot \boldsymbol{k} + F \cdot \boldsymbol{k} = \boldsymbol{0}$$

$$\boldsymbol{M}_S = \sum \boldsymbol{M}_i = F \cdot R \cdot \boldsymbol{i} + F \cdot R \cdot \boldsymbol{i} = 2 \cdot F \cdot R \cdot \boldsymbol{i} = (300\boldsymbol{i})Nm$$

To be able to size and control the beam, the torque moment diagram, the dangerous cross section, the stress distribution and the dangerous points have to be determined (see in figure below).



a, To be able to control the beam for the maximum shear stress, the basic equation of sizing and control of beam structures in the case of pure torsion has to be written.

$$\tau_{max} \leq \tau_{all}$$

For this example the following is valid.

$$\tau_{max} = \frac{|M_c|}{K_p} \leq \tau_{all}$$

The minimum required polar section modulus can be determined.

$$K_{pmin} = \frac{|M_t|}{\tau_{all}} = \frac{|300 \cdot 10^3 \ Nmm|}{60 \ MPa} = 5000 \ mm^3$$

Using the minimum required polar section modulus the minimum required diameter can be determined.

$$K_{pmin} = \frac{D_{min}^3 \cdot \pi}{16} \rightarrow D_{min} = \sqrt[3]{\frac{16 \cdot K_{p_{min}}}{\pi}} = \sqrt[3]{\frac{16 \cdot 5000 \ mm^3}{\pi}} = 29.42 \ mm$$

After rounding up the minimum required diameter we can get the applied diameter.

$$D_{app} = 30mm$$

b, In the case of choosing ring cross section the given diameter ratio should be used. For the calculated torque moment the same minimum required polar section modulus is valid in this case as well. Reordering the polar section modulus for ring cross section using the diameter ratio we get

$$K_p = \frac{({D_2}^4 - {d_2}^4)\pi}{32D_2} = \frac{\left[\left(\frac{9}{8}d_2\right)^4 - {d_2}^4\right]\pi}{32 \cdot \frac{9}{8}d_2} = \frac{\left[\left(\frac{3^8}{4096}{d_2}^4\right) - {d_2}^4\right]\pi}{36 \cdot d_2} =$$

$$= \frac{\frac{2465}{4096}{d_2}^3\pi}{36} = \frac{2465{d_2}^3\pi}{4096 \cdot 36}$$

So the minimum required polar section modulus is

$$K_{pmin} = \frac{2465{d_{2min}}^3\pi}{147456}$$

$$d_{2min} = \sqrt[3]{\frac{147456 \cdot K_{pmin}}{2465\pi}} = 45.66\,mm$$

Using the minimum required inner diameter the applied inner and outer diameters can be determined.

$$d_{2app} = 48mm, \quad D_{2app} = 54mm$$

c, Comparing the weight of the shaft:

$$\frac{m_1}{m_2} = \frac{\rho V_1}{\rho V_2} = \frac{\rho A_1 l}{\rho A_2 l} = \frac{A_1}{A_2} = \frac{\frac{D_{app}^2 \cdot \pi}{4}}{\frac{(D_{2app}^2 - d_{2app}^2)\pi}{4}} = \frac{D_{app}^2}{D_{2app}^2 - d_{2app}^2} = 1.4706$$

The value of the weight reduction:

$$100 \cdot \left(1 - \frac{m_2}{m_1}\right) \approx 36\%$$

Overall, the shaft with ring cross section with the applied diameter is easier then the circular shaft, so it is better and economical to choose for application.

8. COMBINED LOADINGS – UNI-AXIAL STRESS STATE

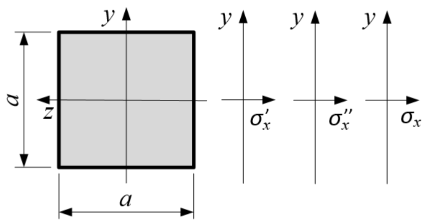
8.1. Examples for the investigations of combined loadings (uni-axial stress state)

Example 1

Combined loading: tension and bending. A prismatic beam with square cross section is loaded and the dangerous cross section is known. The external loading is given with the force and moment vectors reduced to the center of gravity of the cross section.

Data:

$$\boldsymbol{M}_s = (1.4\boldsymbol{k})\ kNm, \boldsymbol{F}_s = (20\boldsymbol{i})\ kN, R_p = 215\ MPa, n = 1.6$$



Questions:

a, Draw the stress distribution on the dangerous cross section and determine the dangerous point(s) of the cross section!

b, Size the beam for maximum stress if the yield point of the material and the factor of safety are known!

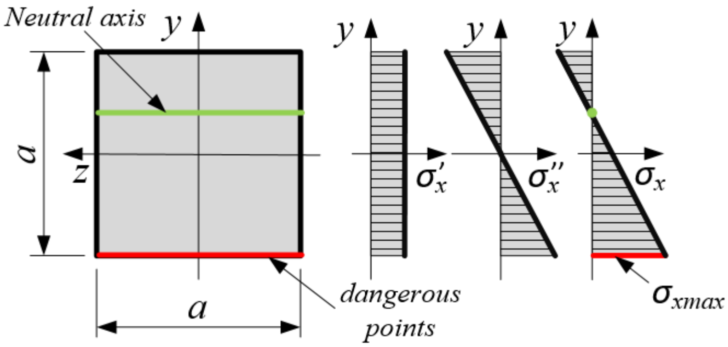
c, Using the applied cross sectional data determine the equation of the neutral axis of the cross section!

Solution:

a, Considering the given force and moment vector reduced to the center of gravity of the cross section the resultant of the beam can be determined. The moment vector has only component in direction z and it is positive meaning that one of the resultant is bending (the bending moment is negative according to the sign convention). The force vector has only component in the direction x and it is positive meaning that the other resultant is tension (the normal force is positive according to the sign convention). In total the resultant of the investigated cross section is combined, tension and bending.

If the resultant is known the stress distribution can be drawn (see in figure below), where σ_x' is the normal stress coming from the tension, while σ_x'' is the normal stress coming from the bending. According to the theorem of superposition the two stress function can be added. The summation of the stress functions result the σ_x stress distribution in the function of y. From the figure it can be seen that the dangerous points of the dangerous cross section are the lower chord of the cross section. It means the maximum stress arises there.

The neutral axis is the set of points on the cross section where the resulting stress value are zero.



b, For the sizing for maximum stress the following condition has to be fulfilled in the case of tension and bending combined loading:

$$\sigma_{xmax} \leq \sigma_{all}$$

The maximum normal stress can be derived from the following equation

$$\sigma_{xmax} = \sigma'_{xmax} + \sigma''_{xmax} = \frac{|N|}{A} + \frac{|M_{bz}|}{K_z}$$

The allowable stress can determined from the following equation

$$\sigma_{all} = \frac{R_P}{n}$$

In the case of tension and bending combined loading the sizing has to be done from the following steps. The first step is to neglect the stress coming from tension and size for only bending. In the second step a control calculation for the combined loading is needed to be performed, where the cross sectional data determined from the bending is used. If the calculated maximum normal stress determined for the suggested cross sectional data is smaller than the allowable stress the control calculation is finished. If the calculated maximum normal stress determined for the suggested cross sectional data is greater than the allowable stress a one bigger standard cross sectional data has to be chosen, then the control calculation is needed to be performed again. This iteration has to be done till the original condition is fulfilled.

We neglect the normal stress coming from the tension and size for only bending, therefore

$$\frac{|M_{bz}|}{K_z} \leq \frac{R_P}{n}$$

From the above equation the minimum required section modulus can be determined

$$K_{zmin} = \frac{|M_{bz}| \cdot n}{R_P} = \frac{1.4 \cdot 10^6 \; Nmm \cdot 1.6}{215 \; \frac{N}{mm^2}} = 10418.60465 \; mm^3$$

While for square cross section the section modulus is

$$K_z = \frac{a^3}{6},$$

therefore

$$K_{zmin} = \frac{a_{min}^3}{6} \rightarrow \; a_{min} = \sqrt[3]{6 \cdot K_{zmin}} = \sqrt[3]{6 \cdot 10418.60465 \; mm^3} = 39.687 \; mm$$

The resulted minimum edge length has to be rounded up, so the applied edge length can be determined

$$a_{app} = 40 \; mm.$$

Control calculation for combined loading using the applied edge length:

The cross sectional properties for the applied edge length

$$A_{app} = a_{app}^2 = 1600 \; mm^2, K_{zapp} = \frac{a_{app}^3}{6} = 10666.67 \; mm^3.$$

The maximum stress in the structure calculated with the applied edge length can be calculated

$$\sigma_{xmax} = \frac{|N|}{A_{app}} + \frac{|M_{bz}|}{K_{zapp}} = \frac{20 \cdot 10^3 \; N}{1.6 \cdot 10^3 \; mm^2} + \frac{1.4 \cdot 10^6 \; Nmm}{1.066 \cdot 10^4 \; mm^3} =$$

$$= 12.5 \; MPa + 131.258 \; MPa = 143.758 \; MPa$$

The allowable stress is

$$\sigma_{all} = \frac{R_P}{n} = \frac{215 \; MPa}{1.6} = 134.375 \; MPa$$

While

$$\sigma_{xmax} > \sigma_{all},$$

the condition is not fulfilled, so the applied edge length calculated for only bending is not enough to bear in the case of combined loading. One bigger standard size has to be chosen, therefore we suggest for applied edge length

$$a_{app} = 45\ mm$$

Control calculation using the newer applied edge length for combined loading:

The cross sectional properties for the newer applied edge length

$$A_{app} = a_{app}^2 = 2025\ mm^2, K_{zapp} = \frac{a_{app}^3}{6} = 15187.5\ mm^3$$

The maximum stress in the structure calculated with the newer applied edge length can be calculated

$$\sigma_{xmax} = \frac{|N|}{A_{app}} + \frac{|M_{bz}|}{K_{zapp}} = \frac{20 \cdot 10^3\ N}{2.025 \cdot 10^3\ mm^2} + \frac{1.4 \cdot 10^6\ Nmm}{1.518 \cdot 10^4\ mm^3}$$
$$= 9.876\ MPa + 92.181\ MPa = 102.057\ MPa$$

While

$$\sigma_{xmax} < \sigma_{all}$$

the condition is fulfilled, therefore the new and standard edge length ($a_{app} = 45\ mm$) is safe to bear the loading.

c, In the case of tension and bending combined loading the equation of the neutral axis can be determined from the following equation:

$$\sigma_x = \frac{N}{A_{app}} + \frac{M_{hz}}{I_{zapp}} \cdot y = 0\ MPa$$

The applied moment of inertia of the cross section is

$$I_{zapp} = \frac{a_{app}^4}{12} = 341718.75\ mm^4$$

Reordering the above equation for the y we get

$$y = -\frac{N}{M_{bz}} \cdot \frac{I_{zapp}}{A_{app}} = -\frac{20 \cdot 10^3}{-1.4 \cdot 10^6} \cdot \frac{3.417 \cdot 10^5}{2.025 \cdot 10^3} = 2.41\ mm$$

The equation of the neutral axis is

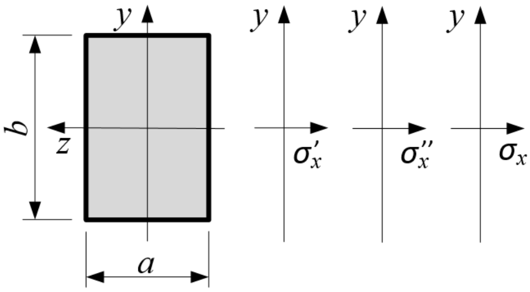
$$y = 2.41\ mm$$

Example 2

Combined loading: compression and bending. A prismatic beam with rectangle cross section is loaded and the dangerous cross section is known. The external loading is given with the force and moment vectors reduced to the center of gravity of the cross section.

Data:

$$\boldsymbol{M}_s = (3\boldsymbol{k})kNm, \boldsymbol{F}_s = (-15\boldsymbol{i})kN, a = 20mm, b = 50mm$$



Questions:

a, Sign the components of the force and moment vector on the cross section and denote the loadings and those signs. Draw the stress distribution of the cross section and determine the dangerous point(s) of the cross section. Draw the neutral axis on the cross section type correctly.

b, Determine the equation of the neutral axis of the cross section!

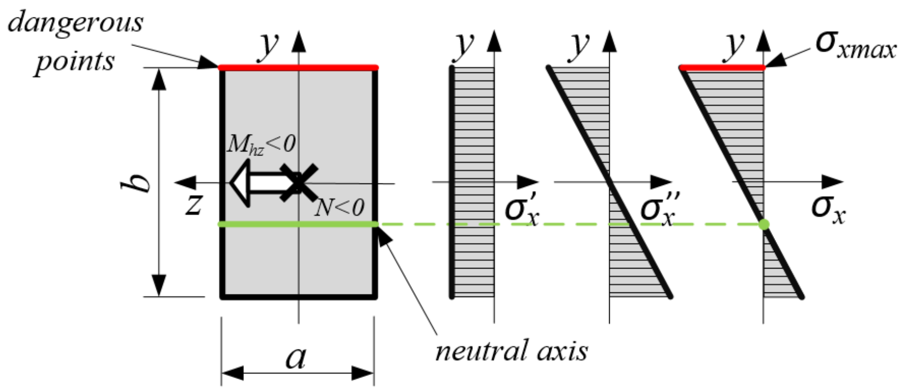
c, Determine the stress tensor connected to the dangerous point(s)!

Solution:

a. Considering the given force and moment vector reduced to the center of gravity of the cross section the resultant of the beam can be determined. The moment vector has only component in direction z and it is positive meaning that one of the resultant is bending (the bending moment is negative according to the sign convention). The force vector has only component in the direction x and it is negative meaning that the other resultant is compression (the normal force is negative according to the sign convention). In total the resultant of the investigated cross section is combined, compression and bending. According to theory of superposition the normal stress can be written as the sum of the normal stress coming from compression and the normal stress coming from bending as follows:

$$\sigma_x' = \frac{N}{A}, \qquad \sigma_x'' = \frac{M_{bz}}{I_z} y, \qquad \sigma_x = \sigma_x' + \sigma_x''$$

If the resultant is known the stress distribution can be drawn (see in figure below), where σ_x' is the normal stress coming from the compression, while σ_x'' is the normal stress coming from the bending. According to the theorem of superposition the two stress function can be added. The summation of the stress functions result the σ_x stress distribution in the function of y. From the figure it can be seen that the dangerous points of the dangerous cross section are the upper chord of the cross section. It means the maximum stress arises there.



b, In the case of compression and bending combined loading the equation of the neutral axis can be determined from the following equation

$$\sigma_x = \sigma_x' + \sigma_x'' = \frac{N}{A} + \frac{M_{bz}}{I_z} y = 0$$

The area and the moment of inertia of the cross section are

$$A = ab = 20mm \cdot 50mm = 10^3 mm^2$$

$$I_z = \frac{ab^3}{12} = \frac{20mm \cdot 50^3 mm^3}{12} = 208333mm^4$$

Reordering the above equation for the y we get

$$y = -\frac{N}{A} \cdot \frac{I_z}{M_{bz}} = -\frac{-15 \cdot 10^3 N}{10^3 mm^2} \cdot \frac{208333mm^4}{-3 \cdot 10^6 Nmm} = -1{,}042mm$$

c, To be able to determine the stress tensor connected to the dangerous points the maximum stress has to be calculated.

The normal stress from the compression is

$$\sigma_x' = \frac{N}{A} = \frac{-15 \cdot 10^3 N}{10^3 mm^2} = -15MPa$$

The normal stress from the bending is

$$\sigma_x'' = \frac{M_{bz}}{I_z} y_{dp} = \frac{-3 \cdot 10^6 Nmm}{208333mm^4} \cdot 25 = -360MPa$$

According to the theory of superposition the maximum stress can be calculated.

$$\sigma_{x_{max}} = \sigma_x' + \sigma_x'' = \frac{N}{A} + \frac{M_{bz}}{I_z} y_{dp} = -15MPa - 360MPa = -375MPa$$

The stress tensor after substitution is

$$\boldsymbol{T}_P = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -360 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$

Example 3

Combined loading: tension and bending. A prismatic beam with ring cross section is loaded with a normal force and a bending moment. The diameter ratio of the ring cross section is also known.

Data:

$$M_{bz} = 100Nm, \quad N = 150kN, \quad \frac{D}{d} = \frac{10}{9}, \quad R_p = 240MPa, \quad n = 1.5$$

Question:

Size the beam for maximum stress, then suggest applied cross sectional sizes considering the diameter ratio!

Solution:

The allowable stress can be calculated as

$$\sigma_{all} = \frac{R_p}{n} = \frac{240MPa}{1.5} = 160MPa$$

In the case of tension and bending the basic equation for sizing can be written as

$$\sigma_{x_{max}} = \sigma_x' + \sigma_x'' = \frac{|N|}{A} + \frac{|M_{bz}|}{K_z} \leq \sigma_{all}$$

While the normal force seems to be more dangerous compared to the bending moment we neglect bending and size for only tension.

$$\left|\frac{N}{A}\right| \leq \sigma_{all}$$

The minimum required area of cross section can be determined

$$A_{min} = \frac{N}{\sigma_{meg}} = \frac{150 \cdot 10^3 N}{160MPa} = 937.5mm^2$$

where the area of cross section for ring cross section can be written as

$$A = \frac{(D^2 - d^2) \cdot \pi}{4}$$

where $D = 5/4d$, therefore

$$A = \frac{\left(\left(\frac{10}{9}d\right)^2 - d^2\right) \cdot \pi}{4} = \frac{\left(\frac{100}{81}d^2 - d^2\right) \cdot \pi}{4} = \frac{\frac{19}{81}d^2 \cdot \pi}{4} = \frac{19d^2 \cdot \pi}{81 \cdot 4}$$

The minimum required inner diameter can be calculated.

$$d_{min} = \sqrt{\frac{81 \cdot 4 \cdot A_{min}}{19 \cdot \pi}} = 71.34mm$$

The minimum required inner diameter has to be rounded up for the following standard value, then the applied inner and outer diameters can be determined.

$$d_{app} = 72mm, D_{app} = 80mm$$

Control calculation for tension and bending combined loading using the applied diameters:

The cross sectional properties using the applied diameters can be determined.

$$A_{app} = \frac{(D^2 - d^2) \cdot \pi}{4} = 955mm^2, K_{zapp} = \frac{(D^4 - d^4)\pi}{32D} = 17286mm^3$$

The maximum stress in the structure using the applied diameters is

$$\sigma_{xmax} = \frac{|N|}{A_{app}} + \frac{|M_{bz}|}{K_{zapp}} = \frac{150 \cdot 10^3 N}{976mm^2} + \frac{0.1 \cdot 10^6 Nmm}{3741 \cdot 10^4 mm^3} = 157.07MPa + 5.78MPa$$

$$\sigma_{x_{max}} = 162.85MPa \leq \sigma_{all} = 160MPa$$

While the condition is not fulfilled one bigger standard cross sectional size has to be chosen.

$$d_{app} = 81mm, D_{app} = 90mm$$

Control calculation for tension and bending combined loading using the newer applied diameters:

The cross sectional properties using the newer applied diameters can be determined.

$$A_{app} = \frac{(D^2 - d^2) \cdot \pi}{4} = 1209mm^2, K_{zapp} = \frac{(D^4 - d^4)\pi}{32D} = 24613mm^3$$

$$\sigma_{xmax} = \frac{|N|}{A_{app}} + \frac{|M_{bz}|}{K_{zapp}} = \frac{150 \cdot 10^3 N}{976mm^2} + \frac{0.1 \cdot 10^6 Nmm}{3741 \cdot 10^4 mm^3} = 124.1MPa + 4.06MPa$$

$$\sigma_{x_{max}} = 128.16MPa \leq \sigma_{all} = 160MPa$$

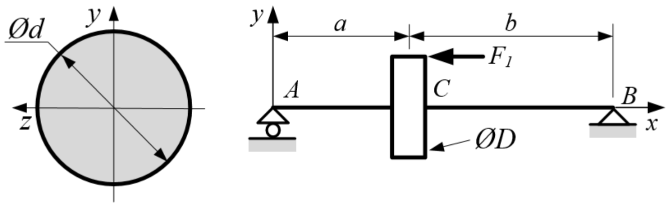
The condition is fulfilled, there the following diameters are suggested for application: $d = 81mm, D = 90mm$.

Example 4

Combined loading: tension and bending. A disc assembled on a circular shaft is loaded by F_1 (see in figure). The stress coming from shear is neglected.

Data:

$$F_1 = 10kN, D = 400mm, a = 400mm, b = 600mm, \sigma_{all} = 160MPa$$



Questions:

a, Reduce the force acting on the disc to the shaft and determine the reaction forces in the supports, furthermore determine the stress resultant diagrams. Determine the dangerous cross section(s) of the shaft and name the resultants. Draw the stress distribution and determine the dangerous point(s) of the cross section!

b, Size the beam for maximum stress and suggest a cross sectional value for application!

Solution:

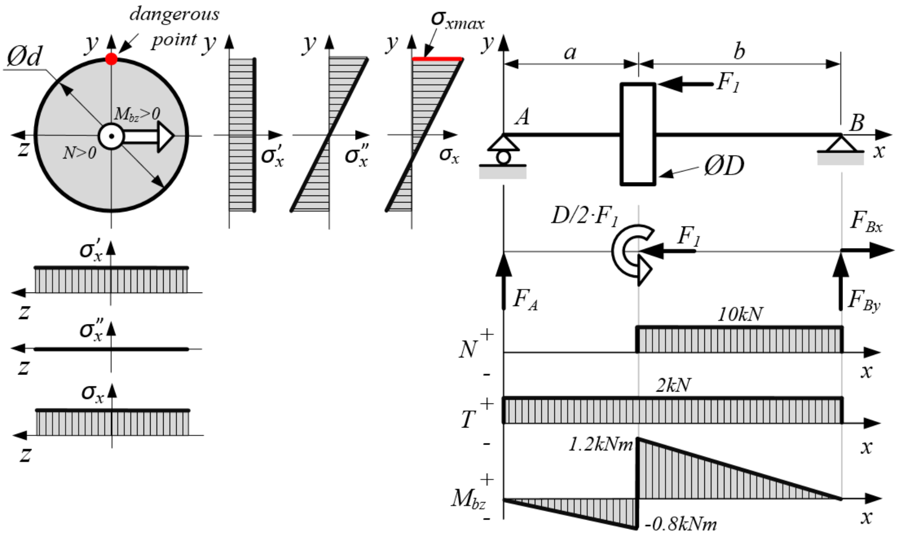
a, Using the equations of equilibrium the reaction forces can be determined:

$$\sum F_{x_i} = 0 = -F_1 + F_{B_x} \Rightarrow F_{B_x} = F_1 = 10kN$$

$$\sum M_{A_i} = 0 = \frac{D}{2} F_1 + (a + b) \cdot F_B \Rightarrow F_B = -\frac{0.2m \cdot 10kN}{1m} = -2kN$$

$$\sum M_{B_i} = 0 = -F_A \cdot (a+b) + \frac{D}{2} F_1 \;\Rightarrow\; F_A = \frac{0.2m \cdot 10kN}{1m} = 2kN$$

To be able to size the shaft the normal force diagram, the shear force diagram, the bending moment diagram, the dangerous cross section, the stress distribution and the dangerous points have to be determined (see in figure below).



The dangerous cross section of the shaft is the cross section C. The resultants are:

$$N = 10kN\,;\; M_{bz} = 1.2kNm\,;\; T = 2\,kN$$

The stress coming from shear is neglected. According to the theory of superposition the normal stresses can be added.

$$\sigma'_x = \frac{N}{A}, \sigma''_x = \frac{M_{bz}}{I_z} y, \sigma_x = \sigma'_x + \sigma''_x$$

b, In the case of tension and bending the basic equation for sizing can be written as

$$\sigma_{x_{max}} = \sigma'_x + \sigma''_x = \frac{|N|}{A} + \frac{|M_{bz}|}{K_z} \le \sigma_{all}$$

As the first step we neglect the normal stress coming from the tension and size for only bending.

$$\frac{|M_{bz}|}{K_z} \le \sigma_{all}$$

The minimum required section modulus can be determined

$$K_{z_{min}} = \frac{|M_{bz}|}{\sigma_{meg}} = \frac{1.2 \cdot 10^6 Nmm}{160MPa} = 7500mm^3$$

Using the minimum required section modulus the minimum required diameter can be calculated.

$$K_z = \frac{d^3\pi}{32} \Rightarrow d_{min} = \sqrt[3]{\frac{32K_{z_{min}}}{\pi}} = \sqrt[3]{\frac{32 \cdot 7500mm^3}{\pi}} = 42.43mm$$

The minimum required diameter has to rounded up for the following standard value, therefore the applied diameter can be determined.

$$d_{app} = 45mm$$

Control calculation for tension and bending combined loading using the applied diameter:

The applied cross sectional properties can be determined.

$$A_{app} = \frac{d^2 \cdot \pi}{4} = 1590mm^2, K_{zapp} = \frac{d^3\pi}{32} = 8946mm^3$$

The maximum stress can be calculated using the applied diameter.

$$\sigma_{xmax} = \frac{N}{A_{app}} + \frac{M_{bz}}{K_{zapp}} = \frac{10^4kN}{1590mm^2} + \frac{1.2 \cdot 10^6Nmm}{8946mm^3} = 6.29 + 134.14MPa$$

$$\sigma_{x_{max}} = 140.43MPa \leq \sigma_{all} = 160MPa$$

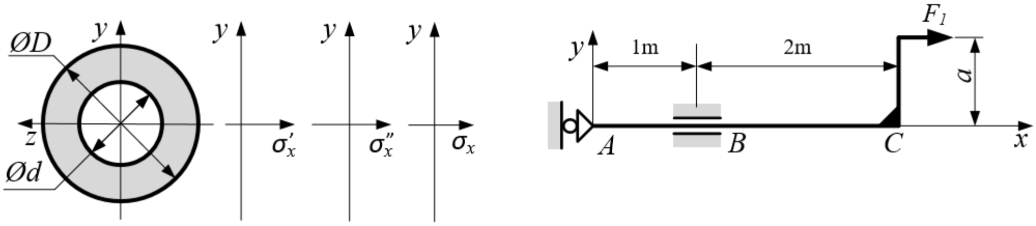
The condition is fulfilled, therefore the application of the diameter $d = 45mm$ is suggested.

Example 5

Combined loading: tension and bending. An off-line beam structure with ring cross section is loaded by a force F_1 (see in figure). The beam is supported with a plain bearings at cross section B.

Data:

$$F_1 = 3kN, \quad \frac{D}{d} = \frac{5}{4}, \quad a = 700mm, \quad \sigma_{all} = 180MPa$$



Questions:

a, Reduce the force to the beam, determine the reaction forces, then draw the stress resultant diagrams. Determine the dangerous cross section(s) of the beam and name the resultants. Draw the stress distribution of the dangerous cross section then determine the dangerous point(s) of the cross section!

b, Size the beam for maximum stress and suggest a cross sectional size for application!

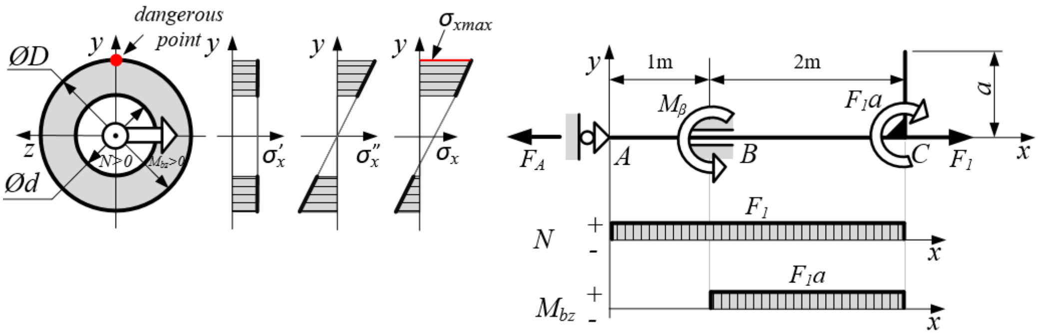
Solution:

a, Using the equations of equilibrium the reaction forces can be determined:

$$\sum F_{x_i} = 0 = -F_A + F_1 \Rightarrow \; F_A = F_1 = 3kN$$

$$\sum M_{A_i} = 0 = \frac{D}{2}F_1 + (a+b)\cdot F_B \;\Rightarrow\; F_B = -\frac{0.2m\cdot 10kN}{1m} = -2kN$$

To be able to size the shaft the normal force diagram, the bending moment diagram, the dangerous cross section, the stress distribution and the dangerous points have to be determined (see in figure below).



The dangerous cross sections of the beam are the section $\overline{BC}$, where the resultants are:

$$N = 3kN\,; \; M_{bz} = a\cdot F_1 = 2.1kNm$$

The stress coming from shear is neglected. According to the theory of superposition the normal stresses can be added.

$$\sigma_x' = \frac{N}{A}, \sigma_x'' = \frac{M_{bz}}{I_z}y, \sigma_x = \sigma_x' + \sigma_x''$$

b, In the case of tension and bending the basic equation for sizing can be written as

$$\sigma_{x_{max}} = \sigma_x' + \sigma_x'' = \frac{|N|}{A} + \frac{|M_{bz}|}{K_z} \leq \sigma_{all}$$

SZÉCHENYI 2020

As the first step we neglect the normal stress coming from the tension and size for only bending.

$$\frac{|M_{bz}|}{K_z} \leq \sigma_{all}$$

The minimum required section modulus can be determined

$$K_{z_{min}} = \frac{|M_{bz}|}{\sigma_{all}} = \frac{2.1 \cdot 10^6 Nmm}{180MPa} = 11667mm^3$$

while $D = 5/4d$, therefore

$$K_z = \frac{(D^4 - d^4)\pi}{32D} = \frac{\left(\left(\frac{5}{4}d\right)^4 - d^4\right)\pi}{40d} = \frac{\left(\frac{625}{256}d^4 - d^4\right)\pi}{40d} = \frac{\frac{369}{256}d^4\pi}{40d} = \frac{369d^3\pi}{256 \cdot 40}$$

Using the minimum required section modulus the minimum required inner diameter can be calculated.

$$d_{min} = \sqrt[3]{\frac{256 \cdot 40K_{z\,min}}{369\pi}} = \sqrt[3]{\frac{256 \cdot 40 \cdot 11667mm^3}{369\pi}} = 46{,}88mm$$

The minimum required diameter has to rounded up for the following standard value, therefore the applied diameters can be determined.

$$d_{app} = 48mm, D_{app} = 60mm$$

Control calculation for tension and bending combined loading using the applied diameters:

The applied cross sectional properties can be determined.

$$A_{app} = \frac{(D^2 - d^2) \cdot \pi}{4} = 1018mm^2, K_{zapp} = \frac{(D^4 - d^4)\pi}{32D} = 12520mm^3$$

The maximum stress can be calculated using the applied diameter.

$$\sigma_{xmax} = \frac{N}{A_{app}} + \frac{M_{bz}}{K_{zapp}} = \frac{3kN}{1018mm^2} + \frac{2.1 \cdot 10^6 Nmm}{12520mm^3} = 2.95MPa + 167.73MPa$$

$$\sigma_{xmax} = 170.68MPa \leq \sigma_{all} = 180MPa$$

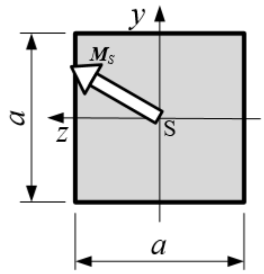
The condition is fulfilled, therefore the application of $d = 48mm$ and $D = 60mm$ are suggested.

Example 6

Combined loading: inclined bending. A prismatic beam with square cross section is loaded and the dangerous cross section is known. The external loading is given with the force and moment vectors reduced to the center of gravity of the cross section.

Data:

$$\boldsymbol{M}_s = (500\boldsymbol{j} + 800\boldsymbol{k})Nm,\ \ \boldsymbol{F}_s = (0\boldsymbol{i})kN,\ \ R_p = 500MPa,\ \ n = 2$$



Questions:

a, Sign the components of the force and moment vector on the cross section and denote the loadings and those signs. Draw the stress distribution of the cross section and determine the dangerous point(s) of the cross section!

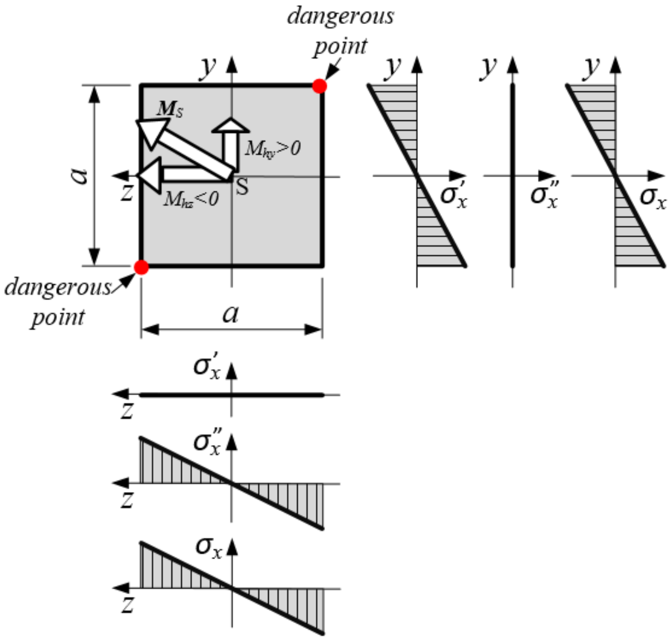
b, Size the beam for maximum stress and suggest a cross sectional size for application!

Solution:

a, According to the theory of superposition the normal stresses can be added.

$$\sigma_x' = \frac{M_{bz}}{I_z}y,\ \ \sigma_x'' = \frac{M_{by}}{I_y}z, \sigma_x = \sigma_x' + \sigma_x''$$

If the resultant is known the stress distribution can be drawn (see in figure below), where σ_x' is the normal stress coming from the bending moment in z direction, while σ_x'' is the normal stress coming from the bending moment in y direction. According to the theorem of superposition the two stress function can be added. The summation of the stress functions result the σ_x stress distribution in the function of y and z. From the figure it can be seen that the dangerous points of the dangerous cross section are two corner points of the cross section. It means the maximum stress arises there.



b, In the case of inclined bending the basic equation for sizing can be written as

$$\sigma_{xmax} = \sigma_x' + \sigma_x'' = \frac{|M_{bz}|}{K_z} + \frac{|M_{by}|}{K_y} \leq \sigma_{all} = \frac{R_p}{n} = 250MPa$$

For square cross section the moment inertias calculated for the direction y and z are the same, so the $K_y = K_z$ are equal to each other.

$$\sigma_{xmax} = \frac{|M_{bz}|}{K_z} + \frac{|M_{by}|}{K_z} = \frac{1}{K_z}\left(|M_{bz}| + |M_{by}|\right) \leq \sigma_{all}$$

The minimum required section modulus can be determined

$$K_{zmin} = \frac{|M_{bz}| + |M_{by}|}{\sigma_{all}} = \frac{|500 \cdot 10^3|Nmm + |-800 \cdot 10^3|Nmm}{250MPa} = 5200mm^3$$

Using the minimum required section modulus the minimum required edge length can be calculated.

$$K_z = \frac{a^3}{6} \Rightarrow a_{min} = \sqrt[3]{6K_{zmin}} = \sqrt[3]{6 \cdot 5200mm^3} = 31.48mm$$

The minimum required edge length has to rounded up for the following standard value, therefore the applied edge length can be determined.

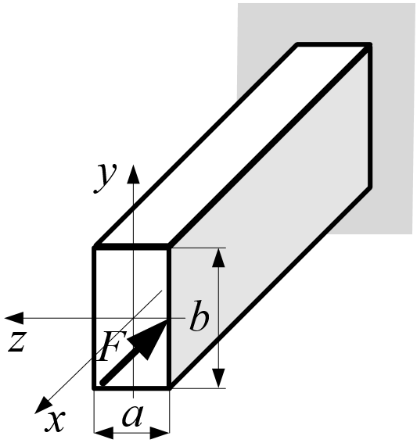
$$a_{app} = 35mm$$

Example 7

Combined loading: tension and bending. A prismatic beam (see in figure) is loaded by an external force F.

Data:

$$F = 90kN, \;\; a = 40mm, \;\; b = 60mm, \;\; \sigma_{all} = 200MPa$$



Questions:

a, Reduce the external force to the center of the axis then determine the resultants and those magnitudes at the support. Determine the dangerous cross section of the beam and draw the stress distribution, then determine the dangerous point(s) of the cross section!

b, Control the beam and determine the actual value of the factor of safety!

c, Determine the equation of the neutral axis!

d, Determine the stress tensor connected to the dangerous point(s) of the cross section!

Solution:

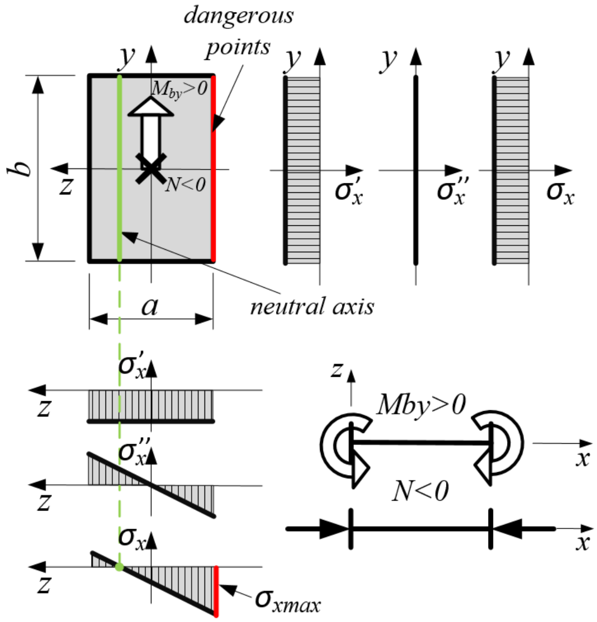
a, For the first step the external force has to be reduced to the centre of gravity of the cross section. The resulting resultants are bending (positive and the direction of the bending moment is y) and compression.

The dangerous cross section of the beam are every cross section, where the resultants are:

$$N = -90kN \, ; \; M_{by} = \frac{a}{2}F = \frac{40mm}{2} 90 \cdot 10^3 N = 1.8 \cdot 10^6 Nmm$$

The stress distribution can be drawn (see in figure below) according to the following contexts using the theory of superposition.

$$\sigma_x' = \frac{N}{A}, \quad \sigma_x'' = \frac{y}{I_y} z, \quad \sigma_x = \sigma_x' + \sigma_x''$$



b, Control calculation for maximum stress using the basic equation for sizing and control as

$$\sigma_{xmax} = \sigma_x' + \sigma_x'' = \frac{|N|}{A} + \frac{|M_{by}|}{K_y} \leq \sigma_{all} = 200MPa$$

The normal stress coming from the compression is

$$\sigma_x' = \frac{|N|}{A} = \frac{|-90 \cdot 10^3 N|}{40 \cdot 60} = 37.5MPa$$

The normal stress coming from the bending is

$$\sigma_x'' = \frac{|M_{by}|}{K_y} = \frac{|1.8 \cdot 10^6 Nmm|}{\frac{a^2 b}{6}} = \frac{|1.8 \cdot 10^6 Nmm|}{16000mm^3} = 112.5MPa$$

The maximum stress using the theory of superposition can be determined.

$$\sigma_{xmax} = \sigma_x' + \sigma_x'' = 37.5MPa + 112.5MPa = 150MPa$$

$$\sigma_{xmax} = 150MPa \leq \sigma_{all} = 200MPa$$

The condition is fulfilled so the beam is right. The actual factor of safety can be calculated as follows

$$n = \frac{\sigma_{all}}{\sigma_{xmax}} = \frac{200MPa}{150MPa} = 1.33$$

c, The equation of the neutral axis can be determined from the following equation

$$\sigma_x = \sigma_x' + \sigma_x'' = \frac{N}{A} + \frac{M_{by}}{I_y} z = 0$$

The cross sectional properties can be calculated.

$$I_y = \frac{a^3 b}{12} = \frac{40^3 mm^3 \cdot 60mm}{12} = 320000mm^4$$

$$A = ab = 40mm \cdot 60mm = 2400mm^2$$

Reordering the equation for the determination of the neutral axis for the z we get

$$z = -\frac{N}{A} \cdot \frac{I_y}{M_{by}} = -\frac{-90 \cdot 10^3 N}{2400mm^2} \cdot \frac{320000mm^4}{1.8 \cdot 10^6 Nmm} = 6.66mm$$

d, To be able to determine the stress tensor connected to the dangerous points the maximum stress has to be calculated.

The normal stress from the compression is

$$\sigma_x' = \frac{N}{A} = \frac{-90 \cdot 10^3 N}{2400mm^2} = -37.5MPa$$

The normal stress from the bending is

$$\sigma_x'' = \frac{M_{by}}{I_y} z_{dp} = \frac{1.8 \cdot 10^6 Nmm}{320000mm^4} \cdot (-20) = -112.5MPa$$

According to the theory of superposition the maximum stress can be calculated.

$$\sigma_{xmax} = \sigma_x' + \sigma_x'' = \frac{N}{A} + \frac{M_{by}}{I_y} z_{dp} = -37.5MPa - 112.5MPa = -150MPa$$

The stress tensor after substitution is

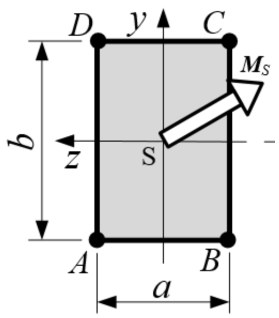
$$\boldsymbol{T}_P = \begin{bmatrix} \sigma_x & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} -150 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$

Example 8

Combined loading: inclined bending. A prismatic beam with rectangle cross section is loaded and the dangerous cross section is known. The external loading is given with the moment vectors reduced to the center of gravity of the cross section.

Data:

$$\boldsymbol{M}_s = (150\boldsymbol{j} - 300\boldsymbol{k})Nm, a = 30mm, b = 50mm$$



Questions:

a, Sign the components of the moment vector on the cross section and denote the loadings and those signs. Draw the stress distribution of the cross section and determine the dangerous point(s) of the cross section!

b, Determine the equation of the neutral axis and draw it along the cross section!

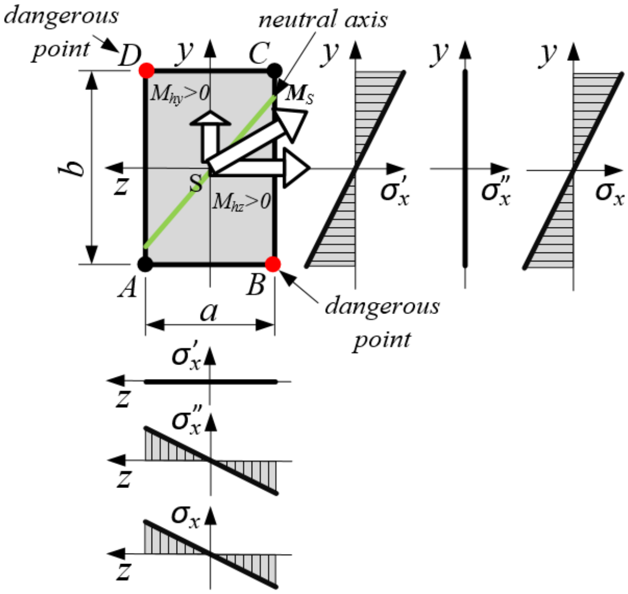
c, Determine the stress values at the given cross sectional points *A, B, C* and *D*!

Solution:

a, According to the theory of superposition the normal stresses can be added.

$$\sigma_x' = \frac{M_{bz}}{I_z} y, \sigma_x'' = \frac{M_{by}}{I_y} z, \sigma_x = \sigma_x' + \sigma_x''$$

If the resultant is known the stress distribution can be drawn (see in figure below), where σ_x' is the normal stress coming from the bending moment in z direction, while σ_x'' is the normal stress coming from the bending moment in y direction. According to the theorem of superposition the two stress function can be added. The summation of the stress functions result the σ_x stress distribution in the function of y and z. From the figure it can be seen that the dangerous points of the dangerous cross section are two corner points of the cross section. It means the maximum stress arises there.



b, The equation of the neutral axis can be written as follows

$$\sigma_x = \sigma_x' + \sigma_x'' = \frac{M_{bz}}{I_z} y + \frac{M_{by}}{I_y} z = 0$$

where the moment of inertias are

$$I_z = \frac{ab^3}{12} = \frac{30mm \cdot 50^3 mm^3}{12} = 312500mm^4$$

$$I_y = \frac{a^3 b}{12} = \frac{30^3 mm^3 \cdot 50mm}{12} = 112500mm^4$$

Reordering the equation for the determination of the neutral axis for the y we get

$$y = -\frac{M_{by}}{I_y} z \cdot \frac{I_z}{M_{bz}} = -\frac{150 \cdot 10^3 Nmm}{112500mm^4} \cdot \frac{312500mm^4}{300 \cdot 10^3 Nmm} = -1.39z$$

c, Normal stress calculation at certain cross sectional points:

Stress value at point A:

$$\sigma_x(A) = \frac{M_{bz}}{I_z} y_A + \frac{M_{by}}{I_y} z_A =$$

$$= \frac{300 \cdot 10^3 Nmm}{312500mm^4}(-25mm) + \frac{150 \cdot 10^3 Nmm}{112500mm^4} 15mm =$$

$$= -24MPa + 20MPa = -4MPa$$

Stress value at point B:

$$\sigma_x(B) = \frac{M_{bz}}{I_z} y_B + \frac{M_{by}}{I_y} z_B =$$

$$= \frac{300 \cdot 10^3 Nmm}{312500 mm^4} (-25mm) + \frac{150 \cdot 10^3 Nmm}{112500 mm^4} (-15mm) =$$

$$= -24MPa - 20MPa = -44MPa$$

Stress value at point C:

$$\sigma_x(C) = \frac{M_{bz}}{I_z} y_C + \frac{M_{by}}{I_y} z_C =$$

$$= \frac{300 \cdot 10^3 Nmm}{312500 mm^4} \cdot 25mm + \frac{150 \cdot 10^3 Nmm}{112500 mm^4} (-15mm) =$$

$$= 24MPa - 20MPa = 4MPa$$

Stress value at point D:

$$\sigma_x(D) = \frac{M_{bz}}{I_z} y_D + \frac{M_{by}}{I_y} z_D =$$

$$= \frac{300 \cdot 10^3 Nmm}{312500 mm^4} 25mm + \frac{150 \cdot 10^3 Nmm}{112500 mm^4} 15mm =$$

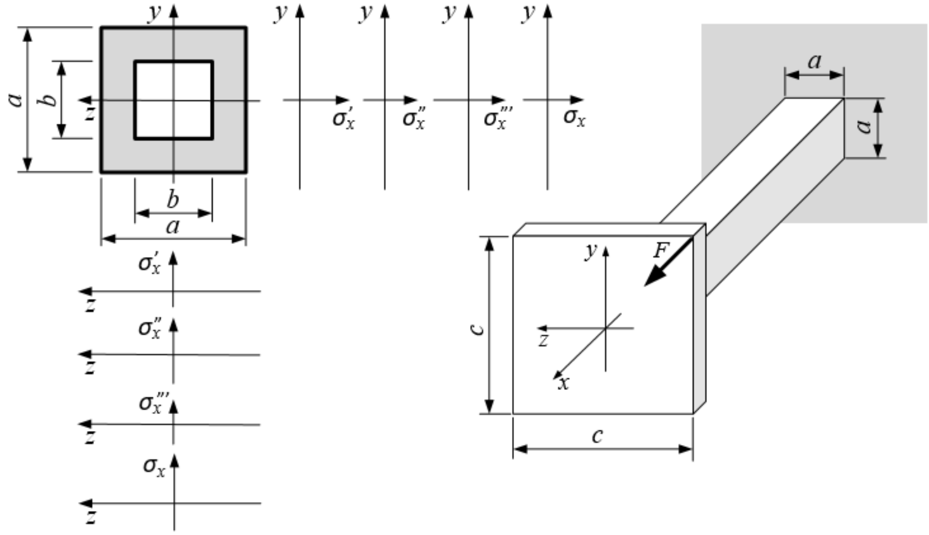
$$= 24MPa + 20MPa = 44MPa$$

Example 9

Combined loading: tension and inclined bending. A sheet welded on a prismatic beam with hollow (rectangular tube) cross section is loaded by an external force F (see in figure).

Data:

$$F = 5kN, \;\; R_p = 350MPa, \;\; n = 1.4, \;\; \frac{b}{a} = \frac{9}{10}, \;\; c = 500mm$$



Questions:

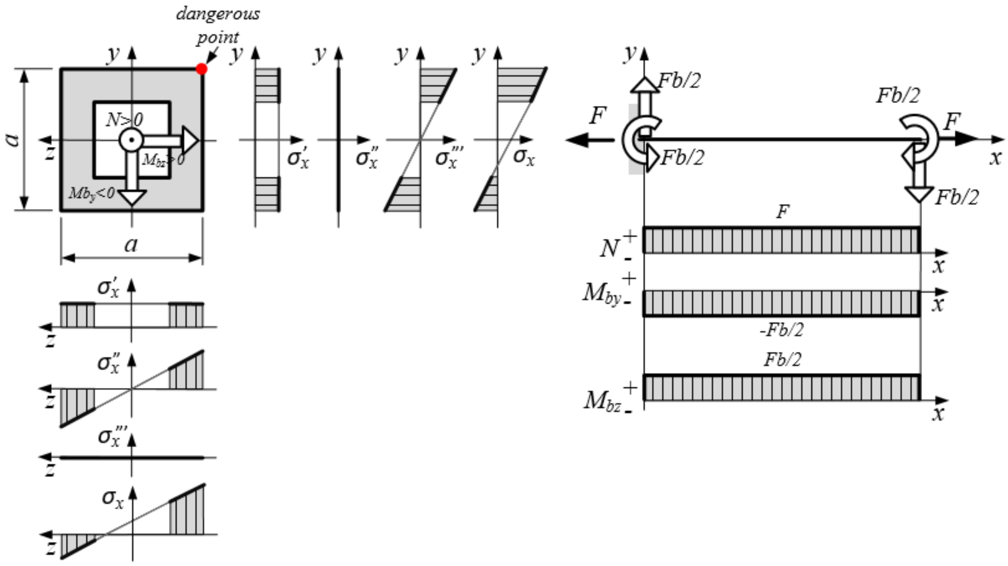
a, Reduce the external force to the center of the axis then determine the resultants and those magnitudes at the support. Determine the dangerous cross section of the beam and draw the stress distribution, then determine the dangerous point(s) of the cross section!

b, Size the beam for maximum stress suggest a cross sectional size for application!

Solution:

a. For the first step the external force has to be reduced to the centre of gravity of the cross section. The resulting resultants are inclined bending and tension.

The stress distribution can be drawn (see in figure below) according to the following contexts using the theory of superposition.



The dangerous cross section of the beam are every cross section, where the resultants are:

$$N = 5kN\ ;\ M_{by} = -\frac{c}{2}F = -1.25kNm; M_{bz} = \frac{c}{2}F = 1.25kNm$$

The normal stresses coming from the different loadings are the following. The theory of superposition can be applied.

$$\sigma_x' = \frac{M_{bz}}{I_z}y, \sigma_x'' = \frac{M_{by}}{I_y}z, \sigma_x''' = \frac{M_{bz}}{I_z}y, \sigma_x = \sigma_x' + \sigma_x'' + \sigma_x'''$$

b, In the case of tension and inclined bending the basic equation for sizing can be written as

$$\sigma_{x_{max}} = \sigma_x' + \sigma_x'' + \sigma_x''' = \frac{|N|}{A} + \frac{|M_{by}|}{K_y} + \frac{|M_{bz}|}{K_z} \leq \sigma_{all}$$

The allowable stress can be determined.

$$\sigma_{all} = \frac{R_p}{n} = \frac{350MPa}{1.4} = 250MPa$$

For square hollow cross section the moment inertias calculated for the direction y and z are the same, so the $K_y = K_z$ are equal to each other.

$$\sigma_{x_{max}} = \frac{|N|}{A} + \frac{|M_{by}|}{K_z} + \frac{|M_{bz}|}{K_z} = \frac{|N|}{A} + \frac{1}{K_z}\left(|M_{bz}| + |M_{by}|\right) \leq \sigma_{all}$$

We neglect the stress coming from the tension and size for inclined bending as follows

$$\frac{1}{K_z}\left(\left|M_{by}\right| + |M_{bz}|\right) \leq \sigma_{all}$$

The minimum required section modulus can be calculated.

$$K_{z_{min}} = \frac{\left|M_{by}\right| + |M_{bz}|}{\sigma_{all}} = \frac{|-1.25 \cdot 10^6|Nmm + |1.25 \cdot 10^6|Nmm}{250MPa} = 10000mm^3$$

Using the minimum required section modulus the minimum required edge length can be determined.

$$K_z = \frac{a^4 - b^4}{6a} = \frac{a^4 - \left(\frac{9}{10}a\right)^4}{6a} = \frac{\frac{10^4 - 9^4}{10^4}a^4}{6a} = \frac{(10^4 - 9^4)a^3}{6 \cdot 10^4} \Rightarrow$$

$$\Rightarrow a_{min} = \sqrt[3]{\frac{6 \cdot 10^4 K_{z_{min}}}{10^4 - 9^4}} = 55.88mm$$

The minimum required edge length has to be rounded up, therefore the applied edge lengths for the hollow cross section can be determined.

$$a_{app} = 60mm, b_{app} = 54mm$$

Control calculation using the applied cross sectional sizes:

The applied cross sectional properties can be determined as

$$A_{app} = a^2 - b^2 = 684mm^2, \;\; K_{z_{app}} = \frac{a^4 - b^4}{6a} = 12380mm^3$$

The maximum stress can be calculated.

$$\sigma_{xmax} = \frac{|N|}{A} + \frac{\left|M_{by}\right|}{K_z} + \frac{|M_{bz}|}{K_z} = \frac{5kN}{684mm^2} + \frac{2 \cdot 1.25 \cdot 10^6 Nmm}{12380mm^3}$$
$$= 7.31MPa + 201.93MPa$$

$$\sigma_{xmax} = 209.24MPa \leq \sigma_{all} = 250MPa$$

The condition is fulfilled, therefore the following cross section is suggested for application: $60mmx60mmx3mm$ square hollow cross section.

9. COMBINED LOADS – TRI-AXIAL STRESS STATE

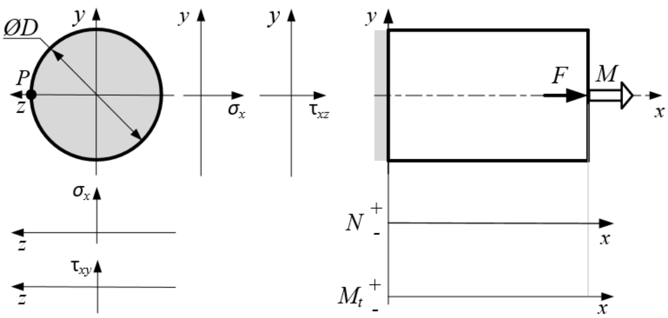
9.1. Examples for the investigations of combined loadings (tri-axial stress state)

Example 1

Combined loading: tension and torsion. A prismatic beam with circular cross section is loaded and the dangerous cross section is known. The external loading is given with the force and moment vectors reduced to the center of gravity of the cross section.

Data:

$$P(0;0;20)mm, F = 150kN, M = 1.3kNm, D = 40mm, G = 80GPa, \nu = 0.3$$



Questions:

a, Determine the reaction forces, then draw the stress resultant diagrams. Determine the dangerous cross section(s) of the beam and name the resultants. Draw the stress distribution of the dangerous cross section then determine the dangerous point(s) of the cross section!

b, Determine the stress values and stress tensor at the given cross sectional point *P*!

c, Establish the strain tensor of body point *P* using the general Hooke's law!

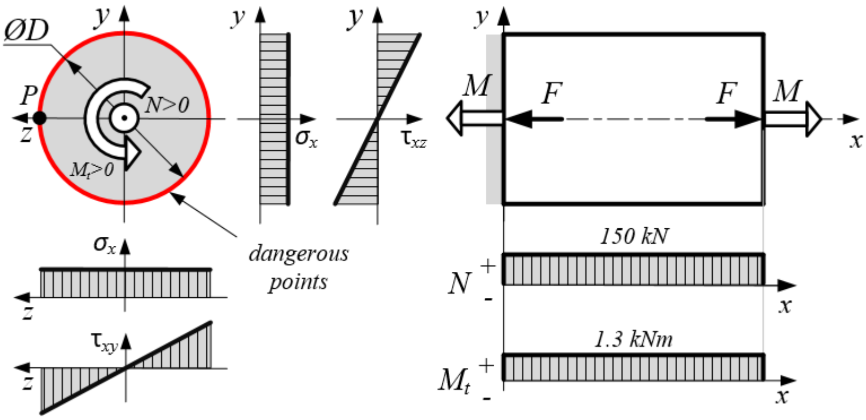
d, Determine the equivalent stress values of body point *P* using the Huber-Mises-Hencky and the Mohr theories for calculations.

e, Edit the Mohr's circle of body point *P* and determine the principal normal stresses, then establish the stress tensor in the coordinate system of principal directions!

f, Calculate the equivalent stress values of body point *P* using principal normal stresses!

Solution:

a, To be able to determine the dangerous points, the normal force diagram, the torque moment diagram, the dangerous cross section and the stress distribution have to be determined (see in figure below).



The dangerous cross sections of the beam are the whole length, where the resultants are:

$$N = 150kN\,;\; M_t = 1.3kNm$$

b, Stress values at the given cross sectional point *P:*

$$\sigma_x(P) = \frac{N}{A} = \frac{150 \cdot 10^3 Nmm}{1256{,}6mm^2} = 119.37MPa$$

$$A = \frac{D^2\pi}{4} = \frac{40^2 mm^2 \cdot \pi}{4} = 1256{,}6mm^2$$

$$\tau_{xz}(P) = \frac{M_t}{I_p} y_p = \frac{1.3 \cdot 10^6 Nmm}{251327{,}41mm^4} \cdot 0mm = 0MPa$$

$$\tau_{xy}(P) = \frac{-M_t}{I_p} z_p = \frac{-1.3 \cdot 10^6 Nmm}{251327{,}41mm^4} \cdot 20mm = -103.45MPa$$

$$I_p = \frac{D^4\pi}{32} = \frac{40^4 mm^4 \cdot \pi}{32} = 251327mm^4$$

The stress tensor in general and after substitution

$$\boldsymbol{T}_P = \begin{bmatrix} \sigma_x & \tau_{xy} & \tau_{xz} \\ \tau_{yx} & 0 & 0 \\ \tau_{zx} & 0 & 0 \end{bmatrix} = \begin{bmatrix} 119.37 & -103.45 & 0 \\ -103.45 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} MPa$$

c, Using the general Hooke's law the strain tensor can be determined.

$$\boldsymbol{A}_P = \frac{1}{2G}\left(\boldsymbol{T}_P - \frac{\nu}{1+\nu}T_I \cdot \boldsymbol{I}\right), T_I = \sigma_x + \sigma_y + \sigma_z = 119.37MPa$$

$$\boldsymbol{A}_P = \frac{1}{2G}\left(\boldsymbol{T}_P - \frac{0{,}3}{1{,}3}119.37MPa \cdot \boldsymbol{I}\right) = \frac{1}{2G}\left(\boldsymbol{T}_P - \begin{bmatrix} 27.55 & 0 & 0 \\ 0 & 27.55 & 0 \\ 0 & 0 & 27.55 \end{bmatrix} MPa\right) =$$

$$= \frac{1}{2G}\begin{bmatrix} 91.82 & -103.45 & 0 \\ -103.45 & -27.55 & 0 \\ 0 & 0 & -27.55 \end{bmatrix} MPa = \begin{bmatrix} 5.74 & -6.47 & 0 \\ -6.47 & -1.72 & 0 \\ 0 & 0 & -1.72 \end{bmatrix} 10^{-4}$$

d, In case of multiaxial loading conditions, equivalent tensile stress can be calculated with the following formula:

$$\sigma_{eq} = \sqrt{[\sigma_x(P)]^2 + \beta[\tau(P)]^2}$$

$$\tau(P) = \sqrt{\left[\tau_{xy}(P)\right]^2 + [\tau_{xz}(P)]^2} = 103.45MPa$$

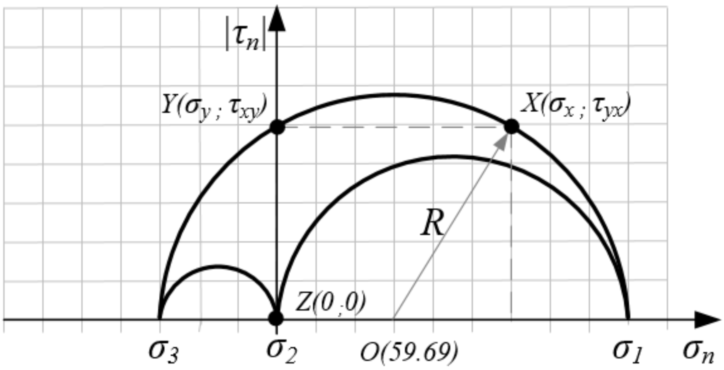
$\beta = 4$ substitution need to be applied when Mohr's theory is used for determining the value of equivalent tensile stress

$$\sigma_{eq}(Mohr) = \sqrt{[\sigma_x(P)]^2 + \beta[\tau(P)]^2} = \sqrt{119.37^2 + 4 \cdot 103.45^2} = 238.87MPa$$

$\beta = 3$ substitution need to be applied when Huber-Mises-Hencky's theory is used for determining the value of equivalent tensile stress

$$\sigma_{eq}(HMH) = \sqrt{[\sigma_x(P)]^2 + \beta[\tau(P)]^2} = \sqrt{119.37^2 + 3 \cdot 103.45^2} = 215.3MPa$$

e, The editing of the Mohr's circle can be seen in figure below.



That circle's radius which passing through points X and Y is determined

$$R = \sqrt{59.69^2 + 103.45^2 = 119.44} = 119.44MPa$$

Using this calculated value the principal normal stresses can be determined

$$\sigma_1 = 179.13MPa;\ \sigma_2 = 0MPa\ ;\ \sigma_3 = -59.75MPa$$

The stress tensor in general and after substitution in the coordinate system of principal directions.

$$\boldsymbol{T}_P(123) = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \\ 0 & 0 & \sigma_3 \end{bmatrix} = \begin{bmatrix} 179.13 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -59.75 \end{bmatrix} MPa$$

f, Value of the equivalent tensile stress using the Mohr's theory for calculations

$$\sigma_{eq}(Mohr) = \sigma_1 - \sigma_3 = 179.13MPa - (-59.75)MPa = 238.88MPa$$

Value of the equivalent tensile stress using the Huber-Mises-Hencky's theory for calculations

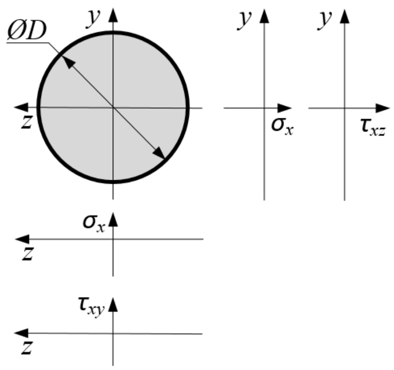
$$\sigma_{eq}(HMH) = \sqrt{\frac{1}{2}[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2]} = 215.3MPa$$

Example 2

Combined loading: tension and torsion. A prismatic beam with circular cross section is loaded and the dangerous cross section is known. The external loading is given with the force and moment vectors reduced to the center of gravity of the cross section.

Data:

$$\boldsymbol{M}_s = (-2\boldsymbol{i})kNm, \boldsymbol{F}_s = (40\boldsymbol{i})kN$$



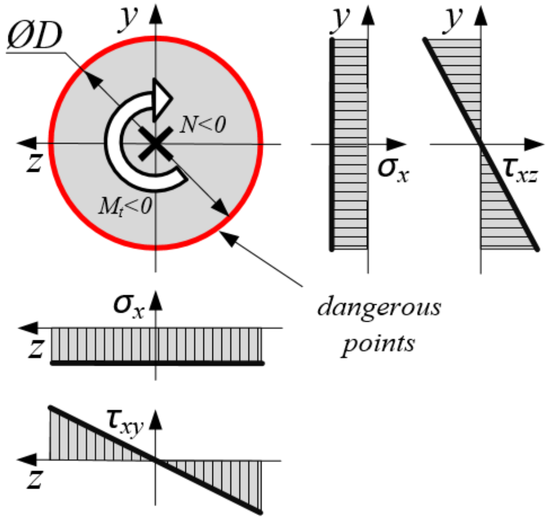
Questions:

a, Sign the components of the force and moment vector on the cross section and denote the loadings and those signs. Draw the stress distribution of the cross section and determine the dangerous point(s) of the cross section!

b, Size the beam using the Huber-Mises-Hencky's theory for equivalent stress and suggest a cross sectional value for application!

Solution:

a, If the resultant is known the stress distribution can be drawn (see in figure below).



From the figure it can be seen that the dangerous points of the cross section are the points of whole outer diameter. It means the maximum stress arises there.

b, In case of multiaxial loading conditions, equivalent tensile stress can be calculated

$$\sigma_{eqmax} = \sqrt{\sigma^2 + \beta\,\tau^2} = \sqrt{\left(\frac{N}{A}\right)^2 + \beta\left(\frac{M_t}{K_p}\right)^2} \leq \sigma_{all}$$

While the torque seems to be more dangerous compared to the normal force, we neglect it and because of the Huber-Mises-Hencky's theory $\beta = 3$ can be applied

$$\sqrt{\beta}\frac{M_t}{K_p} \leq \sigma_{all} \Longrightarrow K_{pmin} = \sqrt{\beta}\frac{M_t}{\sigma_{all}} = \sqrt{3}\frac{|-2\cdot 10^6|}{200MPa} = 17321mm^3$$

Using the minimum required polar section modulus the minimum required diameter can be calculated.

$$K_p = \frac{D^3\pi}{16} \Longrightarrow D_{min} = \sqrt[3]{\frac{16K_{p_{min}}}{\pi}} = \sqrt[3]{\frac{16\cdot 17321mm^3}{\pi}} = 44.52mm$$

The minimum required diameter has to be rounded up for the following standard value, therefore the applied diameter can be determined.

$$D_{app} = 45mm$$

Control calculation for tension and torque combined loading using the applied diameters:

The cross sectional properties using the applied diameter can be determined

$$A_{app} = \frac{D^2 \cdot \pi}{4} = 1590mm^2, K_{papp} = \frac{D^3 \pi}{16} = 17892mm^3$$

The maximum stress in the structure using the applied diameters is

$$\sigma_{eqmax} = \sqrt{\sigma^2 + \beta\,\tau^2} = \sqrt{\left(\frac{N}{A}\right)^2 + \beta\left(\frac{M_t}{K_p}\right)^2} =$$

$$= \sqrt{\left(\frac{40kN}{1590mm^2}\right)^2 + 3\left(\frac{2 \cdot 10^6 Nmm}{17892mm^3}\right)^2} = \sqrt{25.15^2 MPa + 3 \cdot 111.78^2 MPa}$$

$$\sigma_{eqmax} = 195.24MPa \leq \sigma_{all} = 200MPa$$

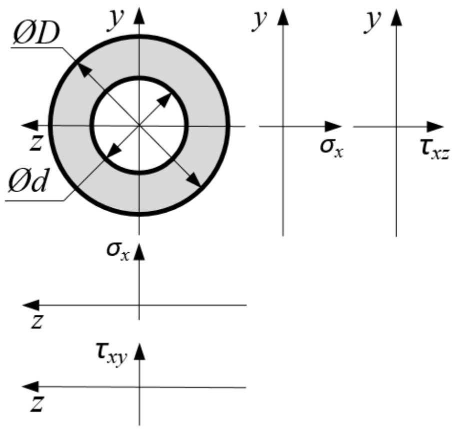
The condition is fulfilled, thereby the following diameter is suggested for application: $D = 45mm$.

Example 3

Combined loading: bending and torsion. A prismatic beam with circular tube cross section is loaded (see in figure below) and the dangerous cross section is known. The external loading is given with the force and moment vectors reduced to the center of gravity of the cross section.

Data:

$$\boldsymbol{M}_s = (-1.6\boldsymbol{i})kNm, \boldsymbol{F}_s = (22\boldsymbol{i})kN, R_p = 260MPa, n = 1.7, D = 3d$$



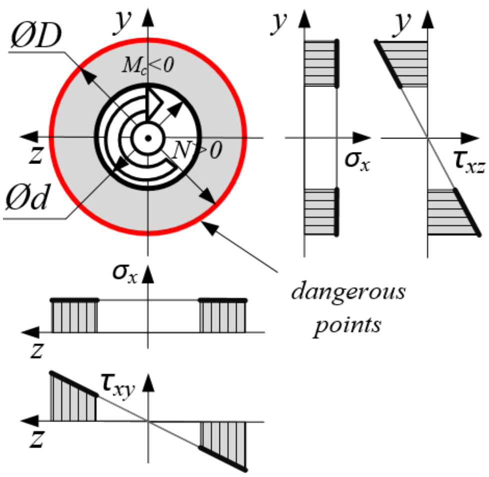
Questions:

a, Sign the components of the force and moment vector on the cross section and denote the loadings and those signs. Draw the stress distribution of the cross section and determine the dangerous point(s) of the cross section!

b, Size the beam using the Mohr's theory for equivalent stress and suggest a cross sectional value for application!

Solution:

a, If the resultant is known the stress distribution can be drawn (see in figure below).



From the figure it can be seen that the dangerous points of the cross section are the points of whole outer diameter. It means that the maximum stress arises there.

b, The allowable stress can be calculated as

$$\sigma_{all} = \frac{R_p}{n} = \frac{260MPa}{1.7} = 152.94MPa$$

In case of multiaxial loading conditions, equivalent tensile stress can be calculated

$$\sigma_{eqmax} = \sqrt{\sigma^2 + \beta\,\tau^2} = \sqrt{\left(\frac{N}{A}\right)^2 + \beta\left(\frac{M_t}{K_p}\right)^2} \leq \sigma_{all}$$

While the torque seems to be more dangerous compared to the normal force, we neglect it and because of Mohr's theory $\beta = 4$ can be applied

$$\sqrt{\beta}\frac{M_t}{K_p} \leq \sigma_{meg} \Longrightarrow K_{pmin} = \sqrt{\beta}\frac{M_t}{\sigma_{all}} = \sqrt{4}\frac{|-1.6 \cdot 10^6|}{152.94MPa} = 20923mm^3$$

while $D = 3d$, therefore

$$K_p = \frac{(D^4 - d^4)\pi}{16D} = \frac{((3d)^4 - d^4)\pi}{48d} = \frac{80d^3\pi}{48}$$

Using the minimum required polar section modulus the minimum required diameter can be calculated.

$$d_{min} = \sqrt[3]{\frac{48K_{p_{min}}}{80\pi}} = \sqrt[3]{\frac{48 \cdot 20923mm^3}{80\pi}} = 15.87mm$$

The minimum required diameter has to be rounded up for the following standard value, therefore the applied diameters can be determined.

$$d_{app} = 16mm, D_{app} = 48mm$$

Control calculation for tension and torque combined loading using the applied diameters:

The cross sectional properties using the applied diameters can be determined

$$A_{app} = \frac{(D^2 - d^2) \cdot \pi}{4} = 1608mm^2, K_{papp} = \frac{(D^4 - d^4)\pi}{16D} = 21447mm^3$$

The maximum stress in the structure using the applied diameters is

$$\sigma_{eqmax} = \sqrt{\sigma^2 + \beta\,\tau^2} = \sqrt{\left(\frac{N}{A}\right)^2 + \beta\left(\frac{M_t}{K_p}\right)^2} =$$

$$= \sqrt{\left(\frac{22kN}{1608mm^2}\right)^2 + 4\left(\frac{1.6 \cdot 10^6 Nmm}{21447mm^3}\right)^2} = \sqrt{13.68^2 MPa + 4 \cdot 74.6^2 MPa}$$

$$\sigma_{eqmax} = 149.83MPa \leq \sigma_{all} = 152.94MPa$$

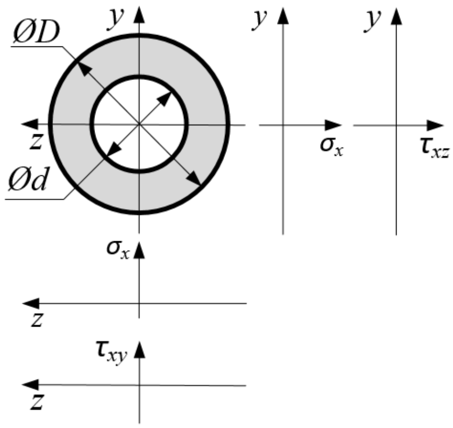
The condition is fulfilled, thereby the following diameters are suggested for application: $d = 16mm, D = 48mm$

Example 4

Combined loading: bending and torsion. A prismatic beam with circular tube cross section is loaded (see in figure below) and the dangerous cross section is known. The external loading is given with the force and moment vectors reduced to the center of gravity of the cross section.

Data:

$$\boldsymbol{M}_s = (-1.4\boldsymbol{i} + 1.2\boldsymbol{j})kNm, \boldsymbol{F}_s = (0\boldsymbol{i})kN, R_p = 285MPa, n = 1.8, D = 3d$$



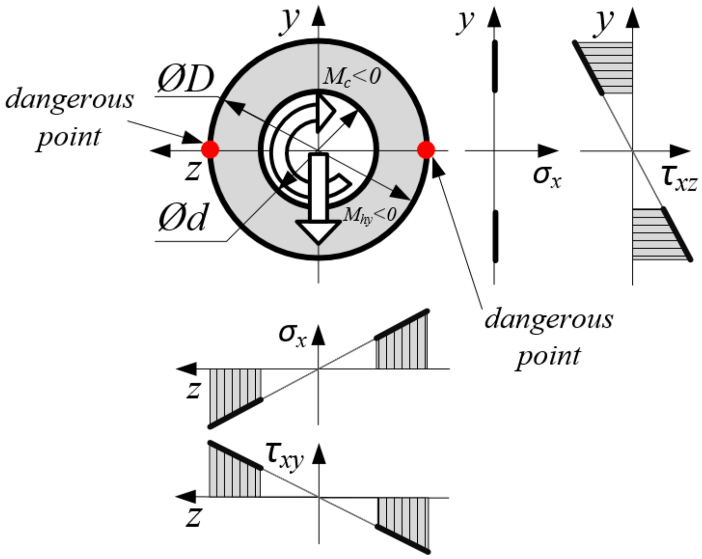
Questions:

a, Sign the components of the moment vector on the cross section and denote the loadings and those signs. Draw the stress distribution of the cross section and determine the dangerous point(s) of the cross section!

b, Size the beam using the Huber-Mises-Hencky's theory for equivalent stress and suggest a cross sectional value for application!

Solution:

a, If the resultant is known the stress distribution can be drawn (see in figure below).



From the figure it can be seen that the dangerous points of the cross section are two outside points of the cross section. It means the maximum stress arises there.

b, The allowable stress can be calculated as

$$\sigma_{all} = \frac{R_p}{n} = \frac{285MPa}{1.8} = 158.33MPa$$

In case of multiaxial loading conditions, equivalent tensile stress is calculated

$$\sigma_{eqmax} = \sqrt{\sigma^2 + \beta\,\tau^2} = \sqrt{\left(\frac{M_{by}}{K_y}\right)^2 + \beta\left(\frac{M_t}{K_p}\right)^2} \le \sigma_{all}$$

Using the $K_p = 2K_z$ substitution because of the circular cross section, furthermore applying the Huber-Mises-Hencky's theory $\beta = 3$ can be applied.

$$\sigma_{eqmax} = \sqrt{\frac{{M_{by}}^2}{{K_y}^2} + \beta\frac{{M_t}^2}{4{K_y}^2}} = \sqrt{\frac{1}{{K_y}^2}\left({M_{by}}^2 + \frac{\beta}{4}{M_t}^2\right)} = \frac{1}{K_y}\sqrt{{M_{by}}^2 + \frac{\beta}{4}{M_t}^2} \le \sigma_{all}$$

$$M_{eq} = \sqrt{{M_{by}}^2 + \frac{\beta}{4}{M_t}^2} = \sqrt{1.2^2kNm^2 + \frac{3}{4}1.4^2kNm^2} = 1.705872kNm$$

$$\frac{M_{eq}}{K_y} \le \sigma_{all} \Longrightarrow K_{ymin} = \frac{M_{eq}}{\sigma_{all}} = \frac{1.705872\cdot 10^6Nmm}{158.33MPa} = 10774mm^3$$

while $D = 3d$, therefore

$$K_{ymin}=\frac{(D^4-d^4)\pi}{32D}=\frac{((3d)^4-d^4)\pi}{32\cdot 3d}=\frac{80d^3\pi}{96}$$

Using the minimum required section modulus the minimum required inner diameter can be calculated.

$$d_{min}=\sqrt[3]{\frac{96K_{ymin}}{80\pi}}=\sqrt[3]{\frac{96\cdot 10774mm^3}{80\pi}}=16.03mm$$

The minimum required diameter has to be rounded up for the following standard value, therefore the applied diameters can be determined.

$$d_{app}=17mm, D_{app}=51mm$$

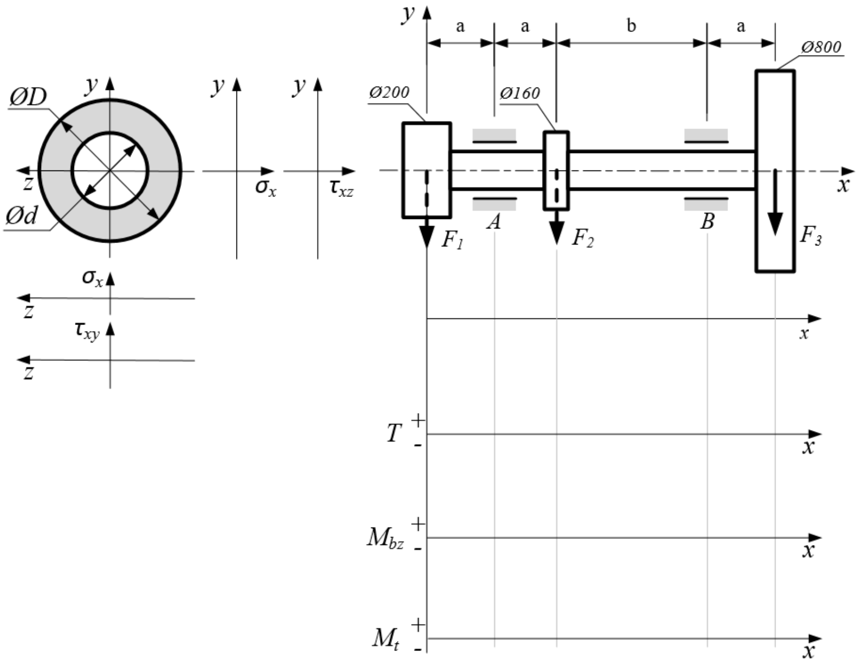
Example 5

Combined loading: bending and torsion. A prismatic beam with ring cross section is assembled with rigid discs. Discs with different diameter are loaded by tangential forces. The bearings at point A and B are not fixing the rotational degree of freedoms. The stress coming from shear is neglected.

Data:

$$F_1=20kN, F_2=25kN, F_3=10kN,$$

$$a=100mm, b=300mm, R_p=260MPa, n=1.5, \frac{D}{d}=\frac{5}{4}$$



Questions:

a, Reduce the forces acting on the discs to the shaft and determine the reaction forces in the supports, furthermore determine the stress resultant diagrams. Determine the dangerous cross section(s) of the shaft and name the resultants. Draw the stress distribution and determine the dangerous point(s) of the cross section!

b, Size the beam using the Huber-Mises-Hencky's theory for equivalent stress and suggest a cross sectional value for application!

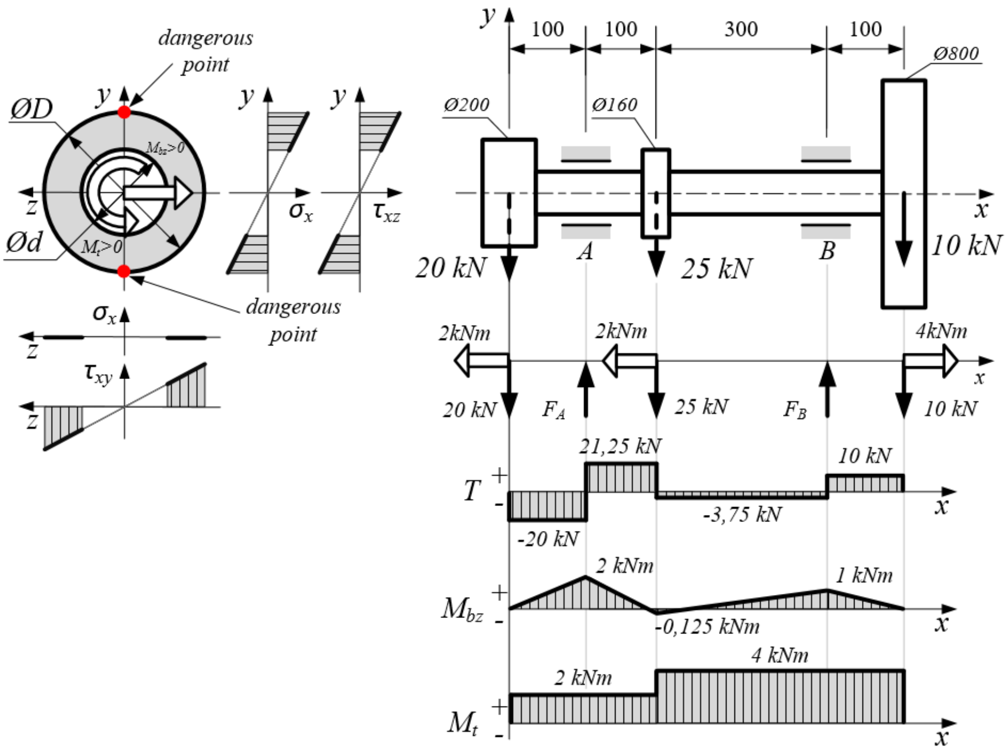
Solution:

a, Using the equations of equilibrium the reaction forces can be determined:

$$\sum M_{A_i} = 0 = F_1 a - F_2 a + F_B(a+b) - F_3(a+b+a) \;\Rightarrow\; F_B = 13.75 kN$$

$$\sum M_{B_i} = 0 = F_1(a+b+a) - F_A(a+b) + F_2 b - F_3 a \;\Rightarrow\; F_A = 41.25 kN$$

To be able to size the shaft the shear force diagram, the bending and torque moment diagrams, the dangerous cross section, the stress distribution and the dangerous points have to be determined (see in figure below).



The dangerous cross section of the shaft is the cross section A or B, further calculations are required to determine it. The resultants are:

Cross section A

$$M_{bz} = 2kNm\,;\ M_t = 2kNm\,;\ T = 21.25\ kN$$

Cross section B

$$M_{bz} = 1kNm\,;\ M_t = 4kNm\,;\ T = 10\ kN$$

The stress distribution and the dangerous points location are the same on the cross section A and B, therefore the procedure for sizing can be equal.

b, The allowable stress can be calculated as

$$\sigma_{all} = \frac{R_p}{n} = \frac{260MPa}{2} = 130MPa$$

In case of multiaxial loading conditions, equivalent tensile stress can be calculated

$$\sigma_{eqmax} = \sqrt{\sigma^2 + \beta\,\tau^2} = \sqrt{\left(\frac{M_{bz}}{K_z}\right)^2 + \beta\left(\frac{M_t}{K_p}\right)^2} \le \sigma_{all}$$

Using the $K_p = 2K_z$ substitution for the ring cross section, furthermore applying the Huber-Mises-Hencky's theory $\beta = 3$ can be applied.

$$\sigma_{eqmax} = \sqrt{\frac{{M_{bz}}^2}{{K_z}^2} + \beta \frac{{M_t}^2}{4{K_z}^2}} = \sqrt{\frac{1}{{K_z}^2}\left({M_{bz}}^2 + \frac{\beta}{4}{M_t}^2\right)} = \frac{1}{K_y}\sqrt{{M_{bz}}^2 + \frac{\beta}{4}{M_t}^2} \leq \sigma_{all}$$

To determine the dangerous cross section, value of M_{eq} need to be calculated

The value of the equivalent moment at cross section A can be calculated

$$M_{eq} = \sqrt{{M_{bz}}^2 + \frac{\beta}{4}{M_t}^2} = \sqrt{2^2 kNm^2 + \frac{3}{4}2^2 kNm^2} = 2.645751 kNm$$

The value of the equivalent moment at cross section B can be calculated

$$M_{eq} = \sqrt{{M_{bz}}^2 + \frac{\beta}{4}{M_t}^2} = \sqrt{1^2 kNm^2 + \frac{3}{4}4^2 kNm^2} = 3.605551 kNm$$

The dangerous cross section of the shaft is the cross section B therefore the sizing needs to be done with this values.

$$\frac{M_{eq}}{K_z} \leq \sigma_{all} \Longrightarrow K_{zmin} = \frac{M_{eq}}{\sigma_{all}} = \frac{3.605551 \cdot 10^6 Nmm}{130 MPa} = 27735 mm^3$$

while $D = 5/4d$, therefore

$$K_{zmin} = \frac{(D^4 - d^4)\pi}{32D} = \frac{\left(\left(\frac{5}{4}d\right)^4 - d^4\right)\pi}{32 \cdot \frac{5}{4}d} = \frac{\left(\frac{625}{256}d^4 - d^4\right)\pi}{40d} = \frac{\frac{369}{256}d^4\pi}{40d} = \frac{369 d^3 \pi}{256 \cdot 40}$$

Using the minimum required section modulus the minimum required inner diameter can be calculated

$$d_{min} = \sqrt[3]{\frac{256 \cdot 40 K_{zmin}}{369\pi}} = \sqrt[3]{\frac{256 \cdot 40 \cdot 27735 mm^3}{369\pi}} = 62.57 mm$$

The minimum required diameter has to be rounded up for the following standard value, therefore the applied diameters can be determined.

$$d_{app} = 64 mm, D_{app} = 80 mm$$

Figures

LITERATURE

[1] MANKOVITS Tamás: *Numerical Analysis of Engineering Structures: Linear Elasticity and the Finite Element Method*. University of Debrecen, Debrecen, 2014.

[2] M. CSIZMADIA Béla, NÁNDORI Ernő: *Szilárdságtan.* Nemzeti Tankönyvkiadó, 1999.

[3] ÉGERT János, JEZSÓ Károly: *Mechanika, Szilárdságtan*. Széchenyi István Egyetem, Győr, 2006.